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Al-Kāshī's Miftāḥ al-Ḥisab, Volume II: Geometry

Translation
and Commentary

 Birkhäuser

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Translation and Commentary

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Preface

Written in 1427 in Arabic by al-Kāshī, *Miftāḥ al-Ḥisab* (Key to Arithmetic, henceforth *Miftāḥ*) is considered as the synthesis and culmination of Islamic¹ progress in mathematics at the time (see [15], p. 22 and [37], p. 66). Luckey’s seminal work [26] on this monumental book has mended the course of research in history of mathematics generally and that of Islamic mathematics in particular [29], rectifying long standing misconceptions regarding important discoveries such as decimal fractions [33], for example. This has profound impact on the study of the evolution of ideas and a fortiori contributes to the edifice of a more inclusive universal history of mathematics. Indeed, due to a chronic lack of study of primary sources in Chinese, Indian², and Islamic civilizations, a Euro-centric narrative of history of science, called “classical narrative” by George Saliba [34] who refutes it through a critical analysis and evidence from primary sources, became dominant. Due to this common narrative, it has been believed for a long time that significant scientific achievements in the Islamic world stopped well before the 15th century. Having received the attention of modern researchers only in the mid-20th century, *Miftāḥ* presents yet another powerful primary-source evidence against many of the claims of the classical narrative. Thus, a full translation of *Miftāḥ* to English to make it accessible to a wide audience has become imperative. A Russian translation of *Miftāḥ* has been available since 1956 [31], and two small sections; one on root extraction [16], another one on measuring the areas of muqarnas [17] have been translated to English. A full translation of *Miftāḥ* to English has been long overdue. This volume on geometry gets us closer to the realization of the full project. It is the second one in the series of three volumes each presenting one of the major areas of mathematics discussed in *Miftāḥ*: Arithmetic [9], Geometry, and Algebra.

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The first author is grateful to his parents Fatima and Ahmed Aydin for all of their sacrifices. He is also grateful to his wife Asiye and children Betül, Beyza, and İsmail for their support and understanding. His special thanks go to Professor Joan Slonczewski of Kenyon College who first came up with an idea that led to his journey into the history of Islamic mathematics that culminated in this publication. He is similarly thankful to Dr. Jennifer Nichols and Dr. Nahla Al-Huraibi who helped him with learning Arabic.

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The third author is particularly grateful to his wife Soumya Bouchachi and his children, Esam, Ibrahim, and Yasmine for being supportive and understanding of his absences and shortcomings. A special dedication to his late parents Benghadfa Hammoudi (1929-2012) and Aicha Messai (1936-2017), may God bless them for their lifelong support of education and pursuit of knowledge.

We dedicate this volume as well as the other two in the series to the memory of Professor Fuat Sezgin, a prominent researcher in the history of mathematics and science in the Islamic civilization. He was a remarkable and leading scholar in the field who was a source of great inspiration for us. His exceptional dedication and monumental work will continue to be important for future researchers. We humbly hope that this volume will contribute to the purpose to which he dedicated his long and productive life.

¹ We use the term Islamic civilization in a general sense to mean the genesis of civilization under the Muslim rule that covers the time period from late 7th century to the 16th century (inclusive) and a large geographic area with much diversity. Scholarly books and treatises in this domain were written in Arabic (the dominant language of scholarship in this civilization) as well as other languages, and authors included both Muslims and non-Muslims (see [9])

² We mean the civilization of all the Indian subcontinent, not just modern day India.

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1 Introduction

1.1 A Biography of al-Kāshī and a Brief History

On March 2, 1427 ([6], p. 32; [17]), Ghiyāth al-Dīn Jamshīd bin Mas'ūd bin Maḥmūd al-Tabīb al-Kāshī (or al-Kāshānī) completed a monumental book in Arabic on arithmetic called *Miftāḥ al-Ḥisāb*. The most commonly used translation of this title, the one which we prefer as well, is "Key to Arithmetic". Other possible translations include "Calculator's Key" and "Key to Calculation". Henceforth we will be referring to it as *Miftāḥ* and its author as al-Kāshī. This encyclopedic work includes three general subjects: arithmetic, geometry, and algebra. It comprises five treatises on arithmetic of integers, arithmetic of fractions, arithmetic of sexagesimal numbers, geometry and measurement, and algebra. Al-Kāshī dedicated the *Miftāḥ* to Ulugh Beg who was the ruler of Samarkand at the time and was a prominent scientist himself. In Berggren's assessment *Miftāḥ* was "the crowning achievement of Islamic arithmetic, and truly a gift fit for a king" ([15], p. 22). Saidan agrees by stating "*Miftāḥ* represents the peak of Islamic arithmetic" ([33], p. 29).

Although we do not know all the details of al-Kāshī's life, through his work and letters to his father, one can affirm that he was born in Kāshān, in modern day Iran some 150 miles south of the capital city Tehran. Therefore, he is also known –perhaps more accurately– as al-Kāshānī but al-Kāshī is more commonly found in the literature. He was born in the later part of the 14th century. Given al-Kāshī's education and well-documented works at hand, some authors suggest 1380 as a plausible year of his birth (e.g. [39], [40], [27], [20]). According to Suter, al-Kāshī died in the year 1436 ([38], [32]). However, based on a note on a manuscript copy of one of al-Kāshī's works, Kennedy proclaims the date of death to be June 22, 1429 (Ramadan 19, 832 A.H.) [42]. This is the commonly accepted date of al-Kāshī's death.

His given name is Jamshīd, his father's name is Mas'ūd and his grandfather's name is Maḥmūd. He has several nicknames best known of which is al-Kāshī. Giyath al-Din means the rescuer of the religion. Al-Tabīb means physician. This is due to the fact that he practiced medicine as a profession before devoting his time totally to the study of mathematics and astronomy later in his life. He is known as al-Kāshī (or al-Kāshānī) because he is from Kāshān. Assuming a nickname based on the place of birth, tribe of origin, or profession was common in the Arabic and Islamic culture. There are many other famous Islamic scholars who are best known by such nicknames such as al-Khwārizmī, al-Tūsī and Omar Khayyam.



Figure 1.1: Iranian stamp featuring an image of al-Kāshī with an astrolabe in the background

During al-Kāshī's early years, Timur (Tamerlaine, 1336-1405) was conquering and ruling vast lands, including central Iran. Widespread turmoil and poverty afflicted the people of the region ([42], [17]). Despite the difficulties at the time, al-Kāshī managed to study science in general, including mathematics and astronomy. In fact, the first dated event that we know of about his life is the observation of a lunar eclipse in his hometown of Kāshān on June 2, 1406, a date recorded in his *Khaqani Zij*¹ as the first date of a series of lunar eclipses he observed. The conditions of the people improved after Shah Rukh Mirza (1377-1447) took over the reign upon his father's death; yet al-Kāshī's best times came only after the young prince Ulugh Beg (1394-1449) became the governor of Samarkand, a city in Transoxiana in modern day Uzbekistan. Ulugh Beg was a recognized scientist himself excelling in mathematics and astronomy who supported many scholars and students. He attracted some of the finest scholars of his time to Samarkand where he first founded a madrasa –a school for advance studies in theology and sciences– between 1417 and 1420, and then establishing the most advanced observatory ever built until that time [24], thus turning Samarkand into a major center of research and learning [6, 7].

We know that al-Kāshī joined the scientific circle of Ulugh Beg upon his invitation around 1420. Al-Kāshī's meticulous writing left us with exact dates and observations of important historical events. Scholars reference several dates for this move, 1417 in ([39], p. 6), 1418 in ([15], p. 21), and 1421 in ([13]). Ulugh Beg's invitation of al-Kāshī came after the latter had proven his scientific ability by accomplishing several significant works between 1406 and the date of his move to Samarkand. On June 2, 1406, al-Kāshī observed and recorded the first of a series of lunar eclipses in Kāshān. In 1407 he completed *Sullam Al-Sama, Ladder of the Heaven on resolution of difficulties met by predecessors in determination of distances and sizes*. In 1410-1411, he wrote *Mukhtasar dar ilm-i hay'at–Compendium of the science of astronomy–* dedicated to Sultan Iskandar, one of the rulers of Timurid dynasty. In 1413-1414, al-Kāshī wrote *Khaqani Zij* and dedicated it to Ulugh Beg. In the introduction of this book al-Kāshī complains about his living conditions of poverty while pursuing important work in mathematics and astronomy. He acknowledges Ulugh Beg's support that allowed him to successfully finish this work. Al-Kāshī sought to attract patronage of a ruler, and this might be his first success. In fact, al-Kāshī did some of his best work in Samarkand under the patronage of Ulugh Beg. This includes *Miftāḥ Al-Ḥisab*, an encyclopedic book on elementary mathematics, and his remarkable approximations to π ([21]) and $\sin(1^\circ)$ ([30]).

1.1.1 Al-Kāshī's Letters: Invaluable Source

Two letters of al-Kāshī in Persian to his father provide us with interesting facts and insights into the scientific environment of Ulugh Beg's court and personalities of some of the important scholars there. From these letters and other sources we learn that Ulugh Beg tolerated al-Kāshī's lack of court etiquette, thanks to his excellent command of mathematics and astronomy [25, 35, 13].

Al-Kāshī's letters were discovered in the second half of the 20th century at different times. The second letter was discovered first and published in 1960 independently by two different scholars. Kennedy gave an English translation of the letter together with a commentary in [25], and Sayili gave both a Turkish translation and an English translation in [35]. The first letter was discovered later by Bagheri who published it with an English translation [13].

Al-Kāshī explains that he wrote the second letter and repeated a lot of information in case the first letter, which was sent via the merchants of Qum, might have been lost. A detailed analysis of the letters suggests that al-Kāshī's father was a learned man who knew mathematics and astronomy. The letters' main topic though is al-Kāshī's professional development, accomplishments, and triumphs, assuring his father of his preeminence that clearly distinguishes him from other scientists in the court. Additionally, he speaks highly of the character of his patron Ulugh Beg, and the progress on the observatory that was being built at the time of the writing of the letters.

1.1.2 Ulugh Beg in al-Kāshī's Letters

The letters depict Ulugh Beg more as an accomplished scientist and scholar who attended many of the scientific meetings held at his court, than as a ruler or statesman. For example, he mentions that Ulugh Beg is well versed in both religious sciences and mathematics. (Translation from [35])

Truth is that, first of all, he knows most of the holy Quran by heart, and he has a ready knowledge of its exegeses. For each occasion he cites an appropriate verse of the Quran, and he makes elegant quotations.

¹ zij is a book of astronomical tables



Figure 1.2: Ulugh Beg's Statue in Samarkand, Uzbekistan (Image source: Wikipedia)

Every day he reads fluently and in the proper manner two sections from the sacred book in the presence of experts who know the whole of the Quran by memory, and no mistakes occur. His knowledge of grammar and syntax is very good, and he writes Arabic extremely well. Likewise, he is well versed in jurisprudence, and he is acquainted with logic and the theory of literary style, as well as with the principles of prosody.

His majesty has great skill in the branches of mathematics. His accomplishment in these matters reached such a degree that one day, while riding, he wished to find out to what day of the solar year a certain date would correspond which was known to be a Monday of the month of Rajab in the year 818 and falling between the 10th and the 15th of the month. On the basis of these data he derived the longitude of the sun to a fraction of 2 minutes by mental calculation while riding on horseback, and when he got down he asked this servant to check his result.

It is true that, as in mental calculation it is necessary to retain quantities in one's mind and to derive others from them, and because in the faculty of memory there is a shortcoming, he [i.e., al-Kāshī himself] could not find the result (correctly) in degrees and minutes and was content with degrees only. But it is not given to any person of our time to do the like; no one else is capable of it.

Ulugh Beg is also described, in these letters, as a kind and open-minded person who is keen on rigorous investigation in science. He listens to all points of views, lets everyone express their opinions, and make their case, and then engages in arguments with students and experts. He allows discussions to continue until the issues become clear to everyone. He does not approve submission to the authority without convincing proofs [35].

He is indeed good-natured to the utmost degree of kindness and charity, so that, at times, there goes on, at the madrasa, between His Majesty and the students of the seeker of knowledge so much arguing back and forth on problems pertaining to any of the sciences that it would be difficult to describe it. He has ordered, in fact, that this should be the procedure, and he has allowed that in scientific questions there should be no agreeing until the matter is thoroughly understood and that people should not pretend to understand in order to be pleasing. Occasionally, when someone assented to His Majesty's view out of submission to his authority, His Majesty reprimanded him by saying "you are imputing ignorance on me." He also poses a false question, so that if anyone accepts it out of politeness he will reintroduce the matter and put the man to shame.

Al-Kāshī reveals that Ulugh Beg is very generous and supports a large number of students seeking knowledge (translation from [13]).



Figure 1.3: Ulugh Beg's Observatory in Samarkand, Uzbekistan. (Image from Wikimedia Commons)

His royal majesty [Ulugh Beg] had donated a charitable gift amounting to 30,000 kopaki dinars, of which 10,000 had been ordered to be given to students. [The names of the recipients] were written down: [thus] 10,000 students steadily engaged in learning and teaching, and qualifying for a financial aid, were listed.

1.1.3 Samarkand in al-Kāshī's letters: Center of Knowledge

The above-mentioned numbers show that Samarkand was a major center of learning at the time. This is supported by further facts mentioned in al-Kāshī's letters. For example, he states that in addition to the students studying with financial aid, there are about 500 persons among notables and their sons who began studying mathematics at twelve different places. He says "there are 24 calculators some of whom are also astronomers and some began [studying] Euclid [’s *Elements*]" [13]. He compares the situation with Kāshān where one or two individuals may be associated with a given discipline, and concludes that the environment in Kāshān is a fortiori not conducive to healthy scientific discussions.

We also learn that serious scientific studies at Samarkand had been undertaken for about twelve years, and that there were at least sixty to seventy mathematicians among Ulugh Beg's staff in addition to astronomers.

Not satisfied with zīj's compiled before his time, Ulugh Beg decided to conduct fresh observations and built the Samarkand observatory. Furthermore, we learn that Ulugh Beg visited Maragheh observatory in his childhood which probably made an impression on him. Given what we know, it would be reasonable to infer that the building of the observatory was planned before al-Kāshī's arrival to Samarkand and started shortly thereafter. Astronomy was one of the most important scientific disciplines in the Islamic tradition ([34]), and observatory was an essential part of that. The significance of observatory in the Islamic world and its legacy in the modern science are summarized by Berggren as "the observatory, as the scientific institution we know today, was born and developed in the Islamic world" ([15], p. 21). The Samarkand observatory that was established by Ulugh Beg turned out to be one of the most important observatories in the Islamic world as stated by Sayili:

From the standpoint of longevity and work therefore the Samarqand Observatory was one of the most important observatories of Islam, and it probably was the most important... Giyāth al-Din Jamshīd and Muin Al-Din-i Kāshī prepared the plan for the observatory... Giyath Al-Din was the first director of the Samarqand observatory ... ([36], p. 265, 266, 271.)

We learn from al-Kāshī's letters to his father that Ulugh Beg followed al-Kāshī's recommendations on certain technical aspects of the building upon being convinced by al-Kāshī's explanations. This led to changes to the original plan during the construction. Al-Kāshī's letters are an invaluable firsthand source of information on the instruments used at the Samarkand observatory.

As stated earlier, a major point al-Kāshī makes in his letters is to reassure his father on al-Kāshī's own elevated status, and prominence among his peers. He conveys that he is the most proficient and competent scientist in

the court. Reading his letters, he might be perceived as boastful, yet we believe it is rather a sign of competence mixed with confidence.

Al-Kāshī narrates how on several occasions other scientists could not solve a given problem even after spending a considerable time on it, yet when he tackles that given problem, he is able to solve it easily. In other instances, when some scholars think that there is a missing information in the problem statement, al-Kāshī shows that this is not the case and solves the problem. Other times, he publicly challenges other scholars on scientific issues and proves his point. He claims this made him famous, and to support his claim he writes of strong words of praise for himself by Ulugh Beg about his character and scientific competency [35].

As to the complimentary remarks of His Majesty, of which mention has been made above, the situation is that no week passes without some friends reporting to this servant that His Majesty made such and such remarks tonight or today concerning me. They are as follows: "He has the knowledge of things ready at hand"; "He knows extremely well"; "His knowledge is superior to that of others"; "His knowledge is more readily at his disposal and more substantial than is the case with Qāḍīzāde"; "His mind works better in this science than that of Qāḍīzāde"; "Mawlana Giyāth al-Dīn knows all the parts of this science, and he solves at once or in a single day a difficulty which takes Qāḍīzāde ten days to disentangle."

He said, likewise, "He is a good and kind-hearted man. All those who have access to our circle, whether they be of the notables or not, have not restrained themselves but have quarreled with people and have transgressed their limits although I have shown them little courtesy. On the other hand, although I have extended much courtesy and many grants to Mawlana Giyath Al-Din and even though he is always honored with access to our company, he has not had any quarrels with anybody and nor has anyone complained about him." "He has not, out of greed, resorted to speak and gossip behind people's backs"; "He conducts himself very well." He has said things of this nature many times.

On the other hand, in describing the majority of scientists in the court, al-Kāshī often refers to their incompetence in disdain [17]. This is certainly a point that he uses to set himself, his patron Ulugh Beg along with Qāḍīzāde al-Rūmī (1364-1436), and very few other scientists apart. Al-Kāshī holds Qāḍīzāde in high esteem and considers him as the most knowledgeable scholar in the group. At the same time, al-Kāshī narrates situations where he excelled better than Qāḍīzāde. An obvious logical consequence of this is that al-Kāshī is the very best scholar in the entire group of Ulugh Beg's scientific staff. He informs his father that he followed his advice to focus on only one subject (in this case it is astronomy) and he understands the reason behind this advice. He stated: "because occupation with another subject would indeed distract me from astronomical observations; second, because of my occupation with another art in which I may be a beginner, there may occur in my discussions or compositions some defect or error which people would bring to bear on the other arts [in which I am adept]" [13]. Evidently, another purpose of al-Kāshī's letters to his father was to counter and refute some rumors his father heard through a person named Shams al-Din. He makes it clear that Shams al-Din's statements about him are in fact false.

Al-Kāshī had a productive time in Samarkand and was the first director of the astronomical observations at the observatory [35]. Unfortunately, he passed away on June 22, 1429 at the observatory in which he was deeply involved, before the full study came to fruition. Qāḍīzāde al-Rūmī took over but sadly, his life was not long enough either to complete the project. Finally, the project was completed under the direction of Ali Qushji (1403-1474).

In the preface of his own zīj written some years after al-Kāshī's death, Ulugh Beg wrote the following testimony for al-Kāshī: "the remarkable scientist, one of the most famous in the world, who had a perfect command of the sciences of the ancients, who contributed to its development, and who could solve the most difficult problems" [42].

In the twentieth century, many scholars recognized al-Kāshī's work in general and *Miftāḥ* in particular. For example, Fuat Sezgin, one of the prominent researchers in the history of Islamic science, summarized it all as:

The achievements of al-Kāshī represent the culmination of Islamic civilization's progress in mathematics. Some of these achievements reappeared or rediscovered later in Europe. ([37], p. 66)

1.2 List of al-Kāshī's Known Works

In addition to *Miftāḥ* which is the most comprehensive work of al-Kāshī, we know of the following works of al-Kāshī. They were collected in *Majmu'* (Collection), Tehran 1888. The following list of al-Kāshī's works can be found in [6] and [42].

- **Zij Khaqani**

Zij' is a Persian word used for astronomy books containing tables that are important for astronomical calculations. This work was written around 1413-1414, before al-Kāshī moved to Samarkand. Written in Persian, it was an update and completion of the *Zij Ilkhani* compiled by famous Nasir al-Din Al-Tusi about 150 years earlier. Bartold believed that al-Kāshī dedicated this work to Shah Rukh who was a patron of science in Herat [14]. However, Kennedy established that the zij was dedicated to Ulugh Beg, son of Shah Rukh and ruler of Samarkand [42]. This may be al-Kāshī's first attempt to secure funding under Ulugh Beg's patronage. This zij contains six treatises and the tables in it are in sexagesimal system [42]. Al-Kāshī used iterative methods to obtain approximate value of the third of each arc [6]. He also states that he collected data from the works of earlier astronomers (astrologers) that did not show up in other tables, along with geometric proofs [6]. Arabic manuscripts are available in London, Oxford, and Istanbul [42].

- **The Treatise on Circumference**

One of al-Kāshī's most remarkable achievements is his approximation of π accurate to 9 sexagesimal places, or 16 decimal places. According to Hogendijk [21], this is one of the highlights of the medieval Islamic mathematical tradition. Before al-Kāshī, the previous record was 6 decimal place approximation by Chinese mathematicians. It took nearly two centuries before the Dutch mathematician Ludolf Van Ceulen surpassed al-Kāshī's accuracy with 20 decimal places. Written in 1424, the original title of this work is *Al-risala al-muhitiyya*. His motivation was to compute π so accurately was to calculate the circumference of the universe so that the round-off error in the approximation would be no more than the width of a horse's hair. He set the accuracy of his approximation in advance and he used a polygon with $3 \cdot 2^{28} = 805,306,368$ sides. Hogendijk states that since al-Kāshī's text was not available in English translation until recently, incorrect or confused statements on the history of π often appeared in the western literature [21]. For example, al-Kāshī was not mentioned in *A History of Pi* [21], a popular book on the subject. Arabic manuscripts are available in Istanbul, Tehran and Meshed [42].

- **The Treatise on Chord and Sine**

Given al-Kāshī's dissatisfaction with earlier zijes, he calculated accurate values for sines and chords in this work to help with astronomical tables. He also calculated $\sin(1^\circ)$ to the same degree of accuracy of his computation of π . Unfortunately, the original text of this work is lost, however, Qāḍīzāde al-Rūmī's account of this treatise is available in the national library of Iran [10] and is translated to Russian as well. The outline of al-Kāshī's main method in this approximation is as follows. He first used the trig identity $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$ which he transformed to a cubic equation of the form $x = a + bx^3$. He then used an iterative method to obtain successive approximations to $\sin(1^\circ)$ that get more accurate at each step. His beautiful approach is still important in modern mathematics and is usually covered under the name of "fixed point iteration" in numerical analysis courses. See [1] for more details on the computation of $\sin(1^\circ)$. There is an edition of this manuscript in *Majmu'* [42].

- **The Zij at-Tashilat (The Zij of Simplifications)**

Al-Kāshī refers to this work in *Miftāḥ* among his works but it is non-existent. It probably included a simplified method of computing the positions of celestial bodies [24].

- **Risala dar Sharh-i Alat-i Rasad (Treatise on the Explanation of Observational Instruments)**

Written for Sultan Iskandar (Kara Yusuf of Black Sheep Turks (Karakoyunlu) dynasty) in 1416, al-Kāshī gives brief yet accurate descriptions of the constructions of eight astronomical instruments. A manuscript copy of this work is available in Leiden, which, according to F. F. Bartold, was written by al-Kāshī himself [6]. Moreover, Persian manuscripts are available in Leiden and Tehran [42].

- **Nuzha Al-Hadaiq (Delight of Gardens)**

Al-Kāshī wrote this treatise in 1416 (February 10, 1416) in Kāshān to which he made additions in Samarkand in 1426. A revised Persian version was written by an anonymous astronomer in Istanbul around 1490 [22].

It gives an explanation of how to build an instrument invented by al-Kāshī which he called “The Plate of Conjunctions” [6]. According to al-Kāshī, it is an instrument from which one can retrieve the calendars of planets, their widths, dimensions, distances from earth, eclipses and other related matters. One can describe this instrument, which has a similar shape as the astrolabe, as the diagram for the approximate graphical solutions of many problems related to the movement of the stars based on the mean values of their coordinates. Al-Kāshī’s later additions include ten appendices which describe additional techniques to utilize the instrument [24]. Arabic manuscripts are available in London, Dublin and Bombay [42].

- **Sullam Al-Sama (The Ladder of the Sky or The Stairway of Heaven)**

al-Kāshī completed this work in his hometown in 1407. It is the earliest known work of al-Kāshī. The full title of this text in the area of astronomy is “on Resolution of Difficulties Met by Predecessors in Determination of Distances and Sizes (of Heavenly Bodies)” [27]. Al-Kāshī mentions this treatise in the preface of *Miftāḥ*. He implies that scholars before him had difficulties and disagreements about distances and sizes of the heavenly bodies. So, he decided to write this book in order to help future scholars. Arabic manuscripts are available in London, Oxford, and Istanbul [42]. There is an Iranian TV series with the same title, the Ladder of the Sky, about al-Kāshī’s life which was broadcast during the month of Ramadan of 2009. Full episodes are available at <http://www.shiasource.com/ladder-of-the-sky>

- **Talkith Al-Miftāḥ (Abridged Key [to Arithmetic])**

Written before the *Miftāḥ* itself (in 1421), it was an early and abbreviated version which contains three treatises and about one-eighth of the material in *Miftāḥ* [10]. Qurbani provided Persian translations of the chapters of *qurbani* [28]. Arabic manuscripts are available in London, Tashkent, Istanbul, Baghdad, Mosul, Tehran, Tabriz, and Patna [42].

- **Miftāḥ al Asbab-fi al-Ilm al-Zīj (The Key of Reason in the Science of Astronomical Tables)**

There is an Arabic manuscript in Mosul (120/306) [42], [32].

- **Risala dar Sakht-i Asturlab (The Treatise on the Construction of an Astrolabe)**

There is a Persian manuscript in Mosul [42].

- **Ta’rib Al-Zīj (Arabization of the Zīj)**

This is an Arabic translation of the introduction of Ulugh Beg’s *zīj* which was in Persian. Manuscripts are available in Leiden and Tashkent [42].

- **Ilḥāqāt an-Nuzha (Supplement to the Excursion, 1427)**

There is an edition of a manuscript in *Majmu’* [42].

- **Wujūh al-Amal al-Darb fi’l-Takht wa’l-Turāb (Ways of Multiplying Using a Dust Board)**

There is an edition of an Arabic manuscript in *Majmu’* [42].

- **Natāij al-Ḥaqāiq (Consequences of Truths)**

There is an edition of an Arabic manuscript in *Majmu’* [42].

- **Risāla fi Ma’rifa Samt al-Qibla min Dāira Hindiyya Ma’rufa (Treatise on the Determination of Direction of the Qibla from a Circle Known as Indian Circle)**

There is an edition of an Arabic manuscript in Meshed (84) [42] [32].

- **Tuḥfat al Sultān fi Asbab al-Irfān (Gift to Sultan on Causes of the Science)**

Oxford (1514). Treatise on astronomy dedicated to Amir-zade Ibrāhīm Sultan, son of Shahruh [32].

- **Risala dar Hay’at (Treatise on Astronomy)**

P-London (Sup. 27261), Yazd (Jami 439/5) [32].

- **Mukhtasar dar Ilm-i Hay’at (Concise Treatise on the Science of Astronomy)**

P-London (869b), Treatise dedicated to Amir-zade Sultan Iskandar Bahadur [32].

- **Risālat ḥall ashkāl Utārid (Treatise on Solution of Propositions on Mercury)**

Meshed (5527) [32].

- **Risala-yi Kamaliyya (Treatise for Kamal al-Din)**
P-Hyderabad (riyada 125-126), Astronomical treatise in 7 books and a conclusion written in Kashan in 1406, dedicated to vizier Kamal al-Din Mahmud [32].
- **Risāla fī Tashīḥ Awsat al-Qamar min al-Arsad al-Khusufiyya (Treatise on Closer Definition of the Center of the Moon from Observations of Eclipses)**
Cairo (riyada 898/23) [32].
- **Zīj al-Kāshī**-Meshed (5321) [32]
- **Notes on Linear Interpolation** -Cairo (Zaki 917/14) [32]

1.3 Manuscript Copies of *Miftāḥ*

Two different typeset and printed books of *Miftāḥ* that include the original Arabic manuscript together with some commentary and explanations are available. The first one is by Ahmad Sa'id Al-Dimirdash and Muhammad Hamdi Al-Hifni al-Shaykh, published in Cairo in 1969 [7] in which commentaries are in Arabic. The other one was published in 1977 in Damascus by Nabulsi [6]. Nabulsi edition has most commentaries in Arabic and some in French as well. It lists the following known manuscript copies of *Miftāḥ* at the time of its publication.

1. Original book of *Miftāḥ Al-Ḥisab*, written by Jamshīd bin Mas'ud bin Mahmūd al-Kāshī March 3rd, 1427 (830 H.), but it is missing.
2. Al Burgandy's manuscript, which is also missing. According to Muhammad Al-Sadiq Al-Arassengi in the Zahiri manuscript, it is written by Abdul Ali Al-Burgandy on Tuesday, 17th day of Dhu Al-Hijja in 889 H., which corresponds to January 5, 1485.
3. The Leiden manuscript, written in 1558 (965 h).
4. The British Museum manuscript, written in 1589 (887 h).
5. The Zahiri manuscript written in 1691 (1102 h).
6. The Leningrad manuscript, written in 1789 (1204 h).
7. The Scientific Library of Prussia manuscript in Berlin, written in 1886 (1303-1304h).
8. The Public Scientific Library in Berlin manuscript (spr 1824)
9. The Berlin Institute for History of Medicine and Science, number 1 and 2.
10. The National Library of Paris, number 5020.

Dimirdash and Muhammad Hamdi edition also mentions the manuscripts 3,4,6-10. They state that they relied on Leiden manuscript [7]. Their list does not include the Zahiri manuscript but includes one that does not appear in the list above, namely, a stone-print copy in Tehran, located in the Tayomrian cabinet number 255-Math. Nabulsi says he used the following manuscripts: Zahiri, London, and Leiden as well as the print book [7] and the Russian translation by Rosenfeld and Youschkevitch published 1956 [31]. The Zahiri manuscript is the main source of his investigation. Nabulsi regards Zahiri manuscript as the oldest known one, gives the following information about it ([6], p. 31) and includes a picture of the first and last pages.

The Zahiri manuscript, number 7795 amongst the collection of books in the Zahiri library in Damascus. It is written on 128 sheets of paper of dimensions 21x12 cm, the handwriting is nice, the titles are written in red, has commentaries, it has eroded from the sides, embroidered with gold (written by Abd Al Ali bin Muhammad Al Burgandi in 889 then copied by Ibn Muhammad Jafar Sadiq Al Arasinji in 1103).

Nabulsi notices that time intervals between the original work and its subsequent manuscripts listed above are 73 years, 31 years, 102 years, 98 years, and 97 years. He concludes that there must be several other manuscripts of *Miftāḥ* that are still missing to this day. The manuscript we are using is not in Nabulsi's list, and it appears to be one of the manuscripts he suspected was missing. This manuscript is in Süleymaniye library in Istanbul in

its Nuruosmaniye branch with new record number 2532 and old record number 2967. It was written in Ramadan 854 H., which corresponds to October 1450. There are several other manuscripts of *Miftāḥ* in Süleymaniye library which are labeled as Atıfendi1719, Esadefendi13175, Fatih5421, Hamidiye883, and Hüsnüpaşa1268. Some of them appear to be incomplete and some of them do not seem to contain a date of completion. The manuscript that we are using, Nuruosmaniye2967, is the oldest one in the collection, and it is complete. Based on available data in the literature, Nuruosmaniye2967 manuscript seems to be one of the oldest, possibly the oldest, available manuscript copy of *Miftāḥ* to this date. Note that the Zahirî manuscript that Nabulsi considers “the oldest known manuscript of *Miftāḥ* so far” was originally written in 889 H. (1484 CE) which is later than the date of Nuruosmaniye manuscript we use (854H-1450 CE). The only reference in the literature to the Nuruosmaniye manuscript that we are aware of is in [33]. The translators of section 5 of treatise 3 of *Miftāḥ* (on root extraction in sexagesimal system) used the following three manuscript copies.

- Princeton University MS ELS 1189
- India Office Arabic MS L756
- University of Leiden MS Or. 185.

1.4 Pedagogical Aspects of *Miftāḥ*

It is clear that *Miftāḥ* was written as a textbook and a practical guide for people in various professions that need arithmetic and elementary mathematics as mentioned by al-Kāshî himself in the introduction:

I solved many problems I was asked by expert mathematicians either for testing me or for their own learning. Some of those problems could not be solved by one of the six algebraic forms. Through these works I acquired a lot of knowledge to solve elementary mathematical problems in the easiest, most beneficial and most efficient ways using clear methods. I wanted to clarify and compile them so that it becomes a guide to anyone interested. Therefore, I wrote this book and collected all that professional calculators need, avoiding tiring length and annoying brevity. I presented rules for most operations in tables to make them accessible for engineers.

It is also clear that *Miftāḥ* is not a theoretical book as the author did not make any effort to prove or justify the algorithms or procedures even by the standards of that time. The title suggests that arithmetic is viewed as the key in solving any problem that requires calculations. In the introduction of *Miftāḥ*, al-Kāshî describes arithmetic as follows: “Arithmetic is the science of rules to determine numerical unknowns from specific known quantities”. Youschkevitch and Rosenfeld summarized the essence of the book by stating:

In the richness of its contents and in the application of arithmetical and algebraic methods to the solution of various problems, including several geometric ones, and in the clarity and elegance of exposition, this voluminous textbook [that is, *Miftāḥ*] is one of the best in the whole of medieval literature; it attests to both the author’s erudition and his pedagogical ability. Because of its high quality the *Miftāḥ* was often recopied and served as a manual for hundreds of years. ([42], p. 256)

Taani picked on the *pedagogical ability* and further studied the content and pedagogy of *Miftāḥ* [39, 40]. His main findings about al-Kāshî’s pedagogy in *Miftāḥ* are what he calls “multiple paths to practice mathematics” and his “exhaustive exclusive classification methods”. Taani’s formulation of “multiple paths” include the following features: multiple definitions, multiple algorithms, multiple formulas, and multiple solutions [40]. He explains that what inspired him to consider investigating multiple paths in al-Kāshî’s pedagogy was student comments. He initially used some excerpts from al-Kāshî in his precalculus class and his goal was to “get data from students about using historical sources in the classroom.” When he administered a questionnaire about the lesson, student comments on al-Kāshî’s method of using multiple methods/approaches to solve a given problem captured his attention and that became a major topic in his dissertation.

As an example of multiple definitions of mathematical concepts, consider these two definitions of division by al-Kāshî in *Miftāḥ*

1. i) In integers, it is partitioning of the dividend in units of the divisor into a number of equal parts so that each share from the divisor is fixed. Such a share is called the quotient.

2. ii) Its general definition is to obtain the number whose ratio to one is the same as the ratio of the dividend to the divisor.

Al-Kāshī often presents multiple methods to perform mathematical calculations. For example, he presents five different ways of performing multiplication of integers. He gives three different formulas to compute the area of a generic triangle, and four different formulas specifically for computing the area of an equilateral triangle. The last treatise of *Miftāḥ* contains a number of word problems and problems in inheritance (determining the share of each heir according to Islamic inheritance laws). He shows multiple ways of solving these problems.

Taani suggests that his work may be used in a classroom in two different ways [40]. The first is directly using the primary text of al-Kāshī will let the students live the experience of discovering mathematics in the 15th century. The second is to follow al-Kāshī in presenting certain topics from multiple perspectives. He comments that the benefits of using multiple paths include: *providing flexibility of thinking, providing opportunities for comparison, creating a network of ideas, and improving creativity* [40].

Therefore, in addition to its profound significance in the history of mathematics due to its content, *Miftāḥ Al-Ḥisab* is also worthy of attention for its pedagogical aspects.

1.5 Possible Future Directions

As mentioned earlier, not only does al-Kāshī not give proofs or justifications for his algorithms and procedures, but he does not give any information about their origins either. It is not clear what parts are his inventions or contributions and what parts are considered established results. A possible future work could be to research the history of the algorithms (such as finding square roots and higher degree roots) described in *Miftāḥ* to determine what exactly the contributions of al-Kāshī might be. On a different direction, it is quite likely that al-Kāshī's work may have influenced mathematicians who came after him. In a recent article [8] we pointed out one such possible connection. We observed that the famous Flemish mathematician Simon Stevin (1548-1620) presented root finding algorithms in his well-known book *L'arithmétique*, ("Arithmetic"), published in 1585. On the surface, Stevin's algorithms look much different from al-Kāshī's. However, after careful examination, we observe that the underlying algorithm is the same, though there are some curious differences in details and implementation (see [8] for more info). To the best of our knowledge, no direct connection between the two mathematicians is known. Given al-Kāshī's influence on Ottoman mathematicians and the education system [23], one might imagine flow of ideas from Central Asia to Europe through the Ottoman land. This is definitely an area that requires further research.

1.6 Notes on Translation and the Purpose of This Work

This work is a translation with commentaries of the book *Miftāḥ Al-Ḥisab* written by al-Kāshī on March 2, 1427 which we started with volume 1 on arithmetic [9]. This is volume 2 on geometry. The upcoming volume 3 will be on algebra. We are using what we believe to be the oldest manuscript so far at hand, namely the Nuruosmaniye2967 manuscript which was originally written in 854H-1450 CE [2].

Our aim is to make these translations available to researchers and students of history of Islamic mathematics alike in a way that best conveys the original meaning of the manuscript. So, we tried to strike a balance between a literal transliteration and a more modern, smooth English translation. We used footnotes and square brackets to explain any ambiguity. Inserted texts from the margins of the manuscript are included in square brackets with footnote indications. Moreover, we underlined some terms to indicate they are further explained in the glossary. We also added al-Kāshī's table of content about geometry to make it easier for the reader to navigate this volume without needing to refer to volume 1 continually. Since Arabic is a language written and read from right to left, some tables have been reverted to read from left to right, except when there are no ambiguities the tables are left as they are. For example, tables and numbers with sexagesimal digits are reverted when degrees, minutes, and seconds are not explicitly mentioned since they are in reverse order in Arabic and English. When the degrees, minutes and seconds are explicitly written with no confusion, we leave the numbers and tables as they are. We separate sexagesimal digits by colons, and we use a semicolon to separate the integer part from the fractional part.

Furthermore, many of the numbers and tables in the manuscript have been checked against several other manuscripts for accuracy. We indicate that in footnotes. Still, we recommend caution when dealing with complex

computations as some differences and ambiguities might arise between different manuscripts or even in different places within the same manuscript as the handwritten letters and numbers might lead to confusions.

Our goal in this volume is to present the raw material of *Miftāḥ Al-Ḥisab* on geometry with some directions that could possibly enrich further studies by students and researchers. We purposely refrained from making any judgments on the nature, origin or the originality of the subject matter found in the *Miftāḥ*. Some of al-Kāshī's work was in fact original, like the law of cosines for example, but others were known to previous authors centuries ago such as many of Euclid's constructions. We leave the critical in-depth historical study to the invested reader to interpret, position, and effectively study al-Kāshī's work in its own right and/or in a historical context.

1.7 The Original Table of Contents of The Fourth Treatise

Fourth Treatise. On Measurements. It contains an introduction and nine chapters.

Introduction. On Defining the Area.

First Chapter. On the area of a triangle and what is related to it. It contains three sections. The first is on defining a triangle and its parts. The second is on the area of a triangle in general and on finding its dimensions. The third is on the area of an equilateral triangle specifically and on finding its dimensions.

Second Chapter. On the area of quadrilaterals and what is related to it. It contains five sections. The first is on definitions. The second is on the area of squares, and rectangles, and on finding their dimensions. The third is on rhombus and kites. The fourth is on rhomboids and trapezoids. The fifth is on concave kites and on obtuse trapezoids.

Third Chapter. On the area of polygons and what is related to it. It contains five sections. The first is on definitions. The second is on their areas in general and finding their dimensions. The third is on what is specific to regular polygons and finding their dimensions. The fourth is on regular hexagons. The fifth is on octagons.

Fourth Chapter. On the area of the circle and its parts. I mean sectors, portions, annulus and other regions, and what is related to them. It contains five sections. The first is on definitions. The second is on the area of a circle, finding the circumference from the diameter, and vice versa. The third is on the area of sectors and portions, and finding their dimensions. The fourth is on the area of surfaces surrounded by circular curves. The fifth is on the table of sine and how to use it.

Fifth Chapter. On the area of other planar surfaces that we did not discuss such as the circular-like, the bow-tie hexagon, the staircase polygon, the star polygons, the curved polygons, and others.

Sixth Chapter. On the area of spherical solids such as cylinders, cones, spheres, and what is related to them. It contains six sections. The first is on definitions. The second is on the area of cylinders. The third is on the area of cones. The fourth is on the area of a sphere and on finding its diameter. The fifth is on the area of a spherical portion, and finding its dimensions. The sixth is on the area of a spherical segment.

Seventh Chapter. On the volume of solids. It contains eight sections. The first is on the volume of the cylinder. The second is on the volume of the cone. The third is on the volume of the frustum. The fourth is on the volume of a remainder of the cone, and [the volume] of a remainder of a conical frustum. The fifth is on the volume of the sphere. The sixth is on the volume of spherical sectors, and spherical portions. The seventh is on the volume of solids with congruent base sides. The eighth is on volume of other solids.

Eight Chapter. On the Ratio of the Volume of a Solid Object to Its Weight.

Ninth Chapter. On the Measurements of Structures and Buildings and it contains three subsections. The first is on the measurement of an arc and a vault. The second is on the measurement of a hallow dome. The third is on the surface area of muqarnas.

في الرابع والسابع **المقالة** الرابعة
في المساحة وهي مشتملة على مقدمة وتسعة ابواب يشتمل عليها
فصول اثنا عشرة فمع تعريف المساحة والاصطلاحات المستعملة فيها



2 The Fourth Treatise: On Measurements

It contains an introduction and nine chapters that are divided into sections. **The introduction contains the definition of area and the terminology used in it.**

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المساحة تحصيل كمية مانع المسح من مثال المسح به او اجزائه او
 كليهما المقياس وهو في الخط فقط مزووض كذراع او قصبه او اسل
 او قدم او اصبع او غير ذلك وفي السطح مربع ذلك الخط المزووض وفي
 الجسم مكعبه وبعض يحسب السطح بالجمع المقياس والاجسام بالكمية
 كما في الكرياس والاثواب بتطيل يكون احد بعدي ذراعاً والابنية
 والاساطين والسقوف في السمات باللبنة والاخر وهما مجسمان يحيط
 بكل واحد منهما ستة سطوح اثنان مربعان متساويان واربعه مستطيلات
 متساويات متشابهات اضلاعها الاطول يساوي ضلع المربع وزوايا
 تتقاطع السطوح بعضها مع بعض قوائم وكذا الاجرام النلكية ككرة الارض
 النقطة هي الاجزائه والخط ماله طول فقط والسطح ماله طول وعض
 لا غير الجسم ماله طول وعض وعنى والمستقيم من الخطوط هو اقصر
 الخطوط وصل بين النقطتين والمستدير منها ما يكون قريباً بالمستدير
 يتصور في بدو النظر انه مستدير والمستوى من السطوح ما يمكن ان
 نخرج في جميع جهاته خطوطاً مستقيمة والمستدير منها ما يمكن ان يقطع
 بسطح مستوي بحيث يكدث فيه دائرة والخطوط المستقيمة المتوازية
 هي التي لا يتلاقى قط ولو اخرجت في الجسمين في غير النهاية وكذلك
 السطوح المستوية ولو اخرجت في جميع الجهات وقد يقال في غير المستقيمة
 والمستوية منها متوازية اذ لم تختلف الابعاد بينها والزوايا المستقيمة
 هي فرضيتان خطين مستقيمين متلاقيين على نقطة واحدة من غير ان يجدا فاذا

يتركازا واما سواه فهو منحنى
 ونسبته المستديرة
 ما يكون قريباً بالمستدير

Introduction

On the area and its terminology

Measurement is determining the measures of what is being measured as an integer multiple of the unit of measurement, or a fraction multiple of it, or both.

The Unit of measurement in linear measurement is a fixed linear unit, like an arms-length¹ or a rod² or Ashal,³ or a foot, or a finger, or others.

In surface measurement, the unit of measurement is the square of that fixed linear unit, and in solids it is the cube of the linear unit. Some people measure the surfaces not in square units, solids not with cubic units like the measure of cloth and dresses, [but rather] with a rectangle, one of whose dimensions is an arm length. They also measure buildings, pillars, and ceilings in structures in adobe and bricks which are two solids each containing six surfaces: Two equal squares and four equal rectangles with similar sides, the larger side being equal to the side of the square. The angles of intersecting faces meet each other at a right angle. Similarly, the celestial bodies are measured by [the size of] earth.

The point is without a part. **The line** is that which has length only. **A surface** only has length and width. **A solid** has length, width, and depth. **A straight line** is the shortest line⁴ between two given points. **A circle** is what is obtained [by compass]. Anything other than that⁵ is a curve. A circle-like shape is⁶ close to being a circle; that looks like a circle to the eye. **A flat surface**⁷ is a surface for which it is possible to draw straight lines in every direction. **A sphere** is a solid such that when it is intersected with a plane, a circle is obtained. **Parallel straight lines** are those that never meet even when extended indefinitely in either direction. Similarly for parallel planes, even when extended in all directions. In cases other than straight and flat (lines and planes), they are called parallel if the distance between them does not change.

A planar angle is the gap between two straight lines intersecting at a single point without being completed (beyond the point of intersection). If

¹ "Diraa" is a unit of measure which is a cubit length.

² "Kasabah" may also mean wooden stick or yardstick which is 12 Arabic feet.

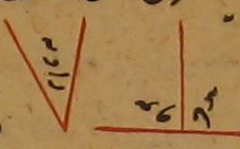
³ In Persian, Ashal means scale and is most likely a Hashimi cubit.

⁴ Path.

⁵ Other than a line and circle.

⁶ Found on the left margin and indicated in the text by an insert.

⁷ Plane.

فاذا افج احد الحطين حدث زاوية اخرى فان كانت مساوية للاولى
 فهي قائمة وان اختلفت فالاصنيق  من القائمة عادة
 والافرع منفرجه  عادة
 الحطين مركزا وادير عليه دائرة فالتوس الموتره بين الحطين من تلك الدائرة
 هي مقدار تلك الزاوية ويقال لما يحدث عن خطين غير مستقيمين زاوية
 ايضا والزاوية المحيطة هي يحدث عن ثلاثة مثلث سطوح مستوية او اكثر
 عند نقطة واحد وكذا ما يحدث عن سطح مستويا واكثر **الباب**
الاول في مساحة المثلث وما يتعلق بها واوردنا فيه ثلثة فصول
الفصل الاول في تعريف المثلث واقسامه المثلث سطح يحيط به
 خطوط مستقيمة يقال لها اضلاع المثلث وعمود المثلث خط مستقيم خارج من
 احدى زواياه قائم على الضلع الموتر لها داخل في المثلث او خارجا و
 يسمى ذلك الضلع بالقاطع مركز المثلث نقطة في سطحه يكون بعدا عن
 جميع الاضلاع متساوية اعني اذا ادير عليها دائرة تماس جميع اضلاعه
 ولهذا سمى بنصف قطر الدائرة الداخلة ولو ان مركز المثلث باكتنته
 هو مركز دائرة احاطت به وتماس زواياه لكنها تحتاج في المساحة بمركز
 الدائرة الداخلة فيه فتسميه بمركز المثلث محازا واما اقسام المثلث
 فتساوي الاضلاع ومتساوي الساقين وقائم الزاوية ومنفرج الزاوية
 وحاد الزوايا هكذا **الفصل الثاني** في مساحة المثلث نعمما و
 استخراج ابعاده بعضها عن بعض واما كيفية مساحة فمن ان نضرب العمود

منفرجة الزاوية
 قائم الزاوية
 منفرجة الزاوية
 منفرجة الزاوية
 منفرجة الزاوية

نصف

one of the two lines is extended, then another angle is formed. If it equals to the first one, then it is a right angle.⁸ If they are different, then the one narrower than a right angle is acute,

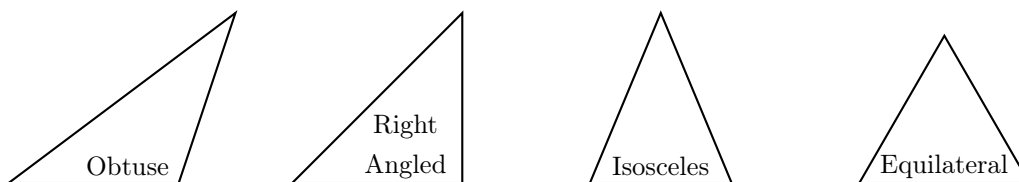


and if wider than a right angle, then it is an obtuse angle. If the intersection of two lines is assumed to be the center and a circle is drawn on it (centered at that point), then the arc of that circle between the lines is the measure of that angle. What is formed from lines that are not straight is called an angle as well. Angles of solid objects are formed from the intersection of three or more flat surfaces at a single point, and similarly for curved surfaces.

First Chapter On the area of a triangle and related matters

We compiled it in three sections.

The First Section is about defining a triangle and its classifications.⁹ A triangle is a surface that is surrounded by three line segments called sides of the triangle. The height of a triangle is a line segment that emanate from an angle and is perpendicular to the opposite side either inside the triangle or outside of it. That side is called the base. The center of the triangle is a point on its surface that is equidistant to all sides of the triangle. I mean, if a circle is drawn around it, it will be tangent to all of the sides. That is why it¹⁰ is called the middle of the diameter of the internal circle.¹¹ Although in reality, the center of the triangle is the center of the circle around it which touches all of its vertices,¹² when calculating the area, we need the center of the incircle of the triangle, so metaphorically, we call it the center of the triangle. As for the types of triangles, they are: equilateral, isosceles, right angled, obtuse angled, and with acute angles like this:



Second Section. On the surface area of a triangle in general and finding some of its dimensions from others. The way to calculate its surface area is to multiply the height

⁸ If the angles are equal to the first right angle, then they are right angles.

⁹ This could also be understood as parts or types, but we choose the word "classification" because in the subsequent sentences, al-Kāshī uses the term "aksamouh" to refer to classifications of triangles.

¹⁰ the center

¹¹ Radius of the incircle

¹² The literal translation here is "angles," but we chose "vertices" because that is what is meant by touching the angles of the triangle.

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على كل ضلع ونضرب
المفضل الثلاثة

في نصف القاعدة أي نسمح العمود والتأ على معايدزاع او غير ذلك من
القياسات ونضرب احدا حاصلين في نصف الآخر **نوع آخر** نضرب العمود
الخارج عن مركز المثلث الى الضلع في نصف جميع الاضلاع ليحصل المساحة
نوع لغز لا يحتاج فيه الى العمود نأخذ فضل نصف مجموع الاضلاع
الثلاثة في احدا الآخرين واكاصل في الآخر واكاصل في نصف مجموع
اضلاع ويحصل جذر حاصل الاخير فهو مساحة المثلث **مثاله**
فرضنا احدا ضلع مثلث عشرة والاخر سبعة وعشر وضلع الباقي احدا
وعشرين فيكون نصف مجموع الاضلاع ٢٤ فضلة على العشرة ١٤
وعلى سبعة عشر ١٤ وعلى احد وعشرين ٣ فنضربنا ١٤ في ٣ حصل ٩
ضربناه في ٣ حصل ٢٩ ضربناه في ٢٤ نصف مجموع الاضلاع حصل
 ٧٧٥٤ احذنا جذره فكان ٨٤ وهو المثلث واما استخراج ابعاد بعضها
عن بعض فمهما استعلام موقع العمود وهو ما يعمل اليه ان نجعل الضلع الا
قاعدة للادولة للظروف وندير على الزاوية التي يوترها الضلع الاطول بعد
الضلع الاقصر دائرة فننصف ما وقع في الدائرة من القاع فهو موقع
العمود ولو اردنا موقع عمود خارج عن زاوية اخرى نجعلها مركزا وندير عليه
بعد احد الضلعين المحيطين بها دائرة فننصف ما وقع في الدائرة من
الضلع الموتر لتلك الزاوية داخل المثلث او خارجا عنه اذا اخرج على
استقامة فهو موقع العمود **مثاله** اردنا ان يحصل موقع عمود خارج
عن زاوية ١ من مثلث ١ على ضلع ٢ جعلنا نقطه ٣ مركزا وادنا

by half of the base.¹³ That is to measure both the height and the base with a yard stick¹⁴ or any other measuring unit and multiply one of the results [of measures] with half of the other.

Another Way. We multiply the height coming out of the center of the triangle on the side (inradius) by half of the sum of the sides to obtain the area.¹⁵

Another way where the height is not needed. We take the difference of half of the sum of the three sides and each side. [Then we multiply one of the three differences]¹⁶ by any of the other [differences], and the result by the last one. [We multiply] the result by half of the sum of the sides. The result of the [square] root of the last result is the surface area of the triangle.¹⁷

Example. We assume one of the sides of a triangle is ten, the other is seventeen, and the last is twenty one. Half of the sum of the sides is 24. The difference with ten is 14 and with seventeen is 7 and with twenty one is 3. We multiply 14 by 7. The result is 98. We multiply it by 3. The result is 294. We multiply it by 24 which is half of the sum of the sides. The result is 7056.¹⁸ Taking its square root we get 84 which is the desired [area].

As for finding dimensions from one another, one goal is to find the placement of the height either¹⁹ by hand [drawing]: We consider the longest side as a base, preferably, but not necessarily. We draw a circle centered²⁰ at the angle [vertex], opposite to the longest side [base] with radius, the shortest side. The midpoint of the arc to the base is the support of the height. If we want (to find) a height coming out of another vertex, we consider it as a center and draw around it a circle with a radius one of the sides that constitute that angle. The midpoint of the arc to the opposite side to that angle, inside the triangle or outside it, if extended, is the location of the height.

Example. We want to find the location of the height coming out of angle A of a triangle ABC on the side BC . We consider the point A as a center and draw around it

¹³ Al-Kāshī is giving the general formula of the surface area of a triangle with base b and height h as $\frac{1}{2}bh$.

¹⁴ Dhiraā is a unit of measure, but we chose to translate it as yard stick as it fits the context.

¹⁵ Al-Kāshī states the formula of the area of a triangle as the in-radius times half the perimeter of a triangle.

¹⁶ Found on the left margin and indicated in the text by an insert symbol.

¹⁷ This is Heron's formula for a triangle with sides a , b , c . The area is

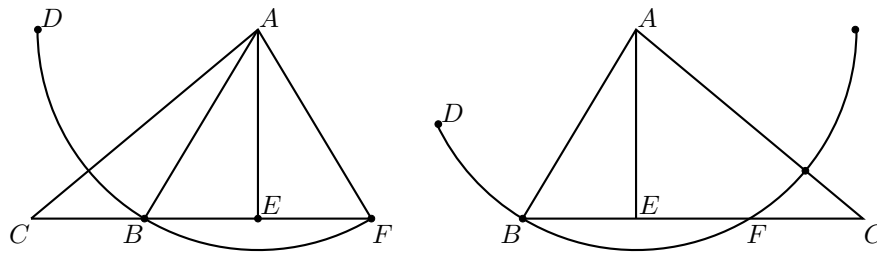
$$\sqrt{\left[\frac{1}{2}(a+b+c)-a\right]\left[\frac{1}{2}(a+b+c)-b\right]\left[\frac{1}{2}(a+b+c)-c\right]\left[\frac{1}{2}(a+b+c)\right]}$$

¹⁸ In the text it is written 7756 but that is not correct as the product is 7056. We believe it is a misprint as the square root in the text is 84 which is the square root of the correct product 7056. Moreover it is correct in the manuscript [3].

¹⁹ Note that al-Kāshī is finding the height either by drawing here, or by computing right after the example he provides.

²⁰ In the graph below, al-Kāshī draws arcs instead of complete circles.

عليها بعدات
 دائرة ط ب د
 ونصفا ب د
 وقع في الدائرة على نقطة وهو موقع العمود فوصلنا آ ه وهو العمود وقع
 في المثلث في الصورة الاولى خارجا عنه في الثانية واما با حساب اذا اردنا
 ان نخرج عن احدى زوايا المثلث عمودا على ضلعين فمجموع الضلعين
 المحيطين بتلك الزاوية في التفاضل بينهما ونقسم الحاصل على الضلع الباقي
 وهو الذي وقع عليه العمود فما خرج ان كان مساويا للضلع الباقي فيكون
 اقصر ذيك الضلعين قائما على القاعدة وان كان اقل منه فوقع العمود
 داخل المثلث وان كان اكثر منه فوقع خارجا عنه ويكون بعد موقعه
 عن ملتقى الضلع الباقى اعني القاعدة مع اقصر الاخرين بقدر نصف التفاضل
 بين القاعدة وخارج القسمة **مثال** فرضنا في مثلث ا ب د ضلع ا ب
 عشرة و **ا ج** سبعة عشر و **ب د** احد او عشرين وارادنا معرفة بعد موقع
 العمود الخارج عن نقطة آ على ضلع ب د من احد طرفيه كان مجموع ا ب
ا د ضربا ه في تفاضلهما وهو **ص** حصل **ا ه** قسما ه على ضلع
 ب د القاعدة وهو **د** خرج من القسمة **ه** ولما كانت اقل من القاعدة
 علم ان العمود وقع داخل المثلث وكون ضلع ب د اطول الاضلاع ل
 عليه ايضا فنقصنا خارج القسمة عن القاعدة وهي **ا د** بقى **ا ه** نصفه
ه وهو بعد موقع العمود عن نقطة ب واعلم ان ضرب مجموع كل عددين



We bisect BF ²¹, which is a chord of the circle, at a point E . It is the location of the height. We draw $\overline{AE}$ which is the height. It is inside the triangle in the first figure and outside the triangle in the second figure. Or by computing: if we want to construct a height on its side, we multiply the sum of the sides forming that angle by their difference. We divide the result on the remaining side which is where the height would be constructed.

Example. We assume in the triangle ABC , the side AB is 10, AC is 17, BC is 21 and we want to know the distance between the position of the height coming out of vertex A , which is perpendicular to BC and to the other end.²² The sum of AB and AC is 27. We multiply it by their difference which is 7, resulting in 189. We divide it by the base BC , which is 21. The outcome of the division is 9. Since it is less than the base, it is known that the height is inside the triangle. BC being the longest of the sides shows that as well. We subtract the result of the division (9) from the base which is 21, 12 remains. Half of it is 6 which is the distance of the position of the height to the point B [i.e., BE]. Know that the product of the sum of two numbers

²¹ Either the arc or the corresponding chord.

²² Looking for the length BE or EC .

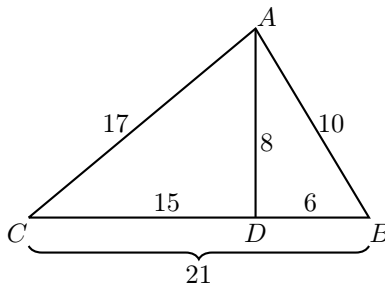
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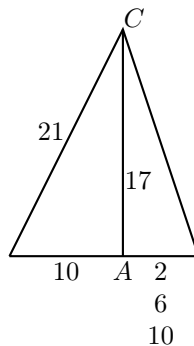
في تفاضلها يساوي تفاضل مربعيها
مثال آخر فان اردنا معرفة موقع العمود
 خارج عن نقطة $\bar{c}$ جمعنا ضلع $\bar{a}$ $\bar{b}$ كان $\bar{c}$ ضربناه في تفاضلها
 وهو $\bar{c}$ حصل $\bar{a}^2$ قسمناه على ضلع $\bar{a}$ وهو $\bar{a}$ خرج $\bar{a}$ ولما
 كان اكثر من قاعد $\bar{a}$ اعلم ان العمود وقع خارج المثلث نقصنا عنه
 ضلع $\bar{a}$ بقي $\bar{a}$ نصفناه صار $\bar{a}$ وهو بعد موقع العمود عن نقطة
 $\bar{a}$ وهو المخط $\bar{a}$ **مثال آخر** يصح منه خارج التفاضل
 شلنا يكون $\bar{a}$ $\bar{b}$ احد اضلاعه وهو $\bar{a}$ عشرة و $\bar{b}$

تسعة و $\bar{a}$ سبعة عشر و اردنا موقع العمود
 الخارج عن نقطة المجمع ضلع $\bar{a}$ $\bar{b}$ كان
 $\bar{c}$ ضربناه في تفاضلها حصل $\bar{a}^2$ قسمناه على قاعد $\bar{b}$ وهي
 $\bar{b}$ خرج من القسمة $\bar{a}$ ولما كان اكثر من ضلع $\bar{b}$ علم ان العمود وقع
 خارجا عن المثلث ونصف التفاضل يكون $\bar{a}$ وهو بعد موقع العمود عن
 نقطة $\bar{b}$ خارجا عنه **طريق آخر** نأخذ التفاضل بين مربع احد الاضلاع
 وبين مجموع مربعي الضلعين الباقيين وننقص احداهما من الضلعين
 قاعد ونقسم نصف التفاضل عليه فما خرج فهو بعد موقع العمود عن
 التي يوترها الضلع الاول ثم ان كان الفضل لمربع الضلع الاول فيكون موقع
 العمود خارجا عن المثلث من جانب هذه الزاوية وان لم يكن التفاضل قبلك
 الزاوية فائمه وان كان الفضل لمجموع الربيعين يكون نصف التفاضل اقل

by their difference equals the difference of their squares.²³



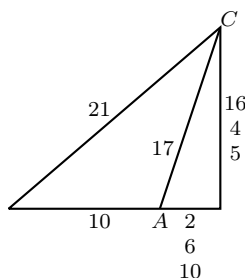
Another Example. If we want to know the location of the height coming out of the point C , we add the sides AC and BC and it is 38. Multiplying it by their difference, which is 4, the result is 152. Dividing it by side AB , which is 10, results in $\frac{15}{5}$. Since it is more than the base AB , know that the height is outside the triangle. We subtract the side AB from it, the result is $\frac{5}{5}$. Half of it is $\frac{2}{10}$ which is the distance of the position of the height from the point A , which is the wanted.²⁴

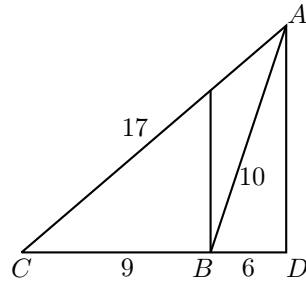


Another example in which the result of the division is an integer. We assume a triangle ABC with sides, AB 10, BC 9, and AC 17. We want the location of the height coming out of the vertex A . The sum of the sides AB and AC is 27, which we multiply by their difference, resulting in 189. We divide it by the base BC which is 9, resulting in 21. Since it is greater than the side BC , it is known that the height is located outside the triangle. Half of the difference is 6 which is the distance between the height and the point B going out of the triangle.

²³ Al-Kāshī is stating the difference of two squares identity: $(a + b)(a - b) = a^2 - b^2$.

²⁴ This figure seems to be incorrect. We correct it here. Note that it is correct in manuscripts [4] and [5].





Another Method. We take the difference between the square of one of the sides and the sum of the squares of the remaining two sides. We assume one of these remaining sides a base and we divide half of the difference by it. What results is the distance of the location of the height from the angle opposite to the first side. Then, if the square of the first side is larger [the difference is positive], then the location of the height is outside the triangle from the side of that angle. If there is no difference, then that angle is a right angle. If the difference for the sum of the squares is positive, then half of the difference is smaller

من مربع القاعدة فوق العمود داخل المثلث وان كان مساويا لزاوية
 التي يحيط بها الضلع الاول مع القاعدة قائمة وان كان اكثر فالعمود
 وقع خارجا عن هذه الزاوية لكن الخارج من القسمة يكون بعد موقع العمود
 عن الزاوية التي يوترها الضلع الاول ولهذا يكون ح اكثر من القاعدة **مثال**
 من المثلث المتقدم كان مربع ضلع ٢٨٩٦١ فنقصنا عنه مجموع مربعي
 الآخرين وهو ١٨١ بقى ١٥٨ ولما كان الفضل طبع الضلع الاول علم
 ان العمود وقع خارجا عن جانب زاوية ب فقسنا نصفه وهو ٩٠ على
 ضلع ب وهو ٦ خرج من القسمة ٩ وهو بعد موقع العمود عن نقطة ب
مثال اخر نقصنا مربع ا ب وهو ١٥٥ عن مجموع مربعي الآخرين وهو ٣٧٥
 بقى ٢٢٥ قسنا نصفه وهو ١٣٩ على القاعدة وهي ٩ خرج من القسمة ٢٥
 وهو بعد موقع العمود عن نقطة د الجانب ب مجاوزا عنه الى الخارج وذلك
 لان نصف فضل مجموع المربعين كان اكثر من مربع القاعدة فاذا نقصنا
 القاعدة عنه بقى البعد عن نقطة ب وهو المراد **والاخر** ان تنقص
 مربع احد الاقصرين من مجموع مربعي الآخرين وتنقسم نصف الباقي على الاطول
 فما خرج فهو بعد موقع العمود على الاطول من طرف الاقص الاخر داخل
 المثلث او ضرب مجموع الاقصرين في تفاضلهما وتنقسم الحاصل على الاطول
 فما خرج من القسمة تنقصه عن الاطول فنصف الباقي هو بعد موقع العمود
 عن طرف الاقص الاضلاع الواقعة على الاطول داخل المثلث ومنها
 معرفة مقدار العمود ضرب بعد موقع العمود عن احد طرفي القاعدة

than the square of the base, and the location of the height is inside the triangle. If they are equal, then the angle between the first side and the base is a right angle. If it is greater, then the height is located outside of that vertex and the result of the division is the distance between the location of the height and the vertex which is opposite to the first side. That is why BC is bigger than the base.

Example. From the previous triangle, the square of the side AC is 289. We subtract from it the sum of the other two squares, which is 181. The result is 108. Since the first square is bigger, it is known that the height is located outside of the vertex B . So we divide its half, 54, by the side BC which is 9 resulting in 6 which is the distance between the location of the height and the point B .

Another Way. We subtract the square of AB which is 100 from the sum of the other two squares, which is 370. The result is 270. We divide its half, which is 135, by the base which is 9 and we get 15 which is the distance between the location of the height and the point C to the side of B extended outside. That is because half of the difference between the sum of the two squares is greater than the square of the base. If we subtract the base from it, then the distance from the point B is 6. This is what we wanted. In another way, we subtract the square of one of the shorter sides from the sum of the squares of the other two sides, and we divide half of the remaining by the longer. What results is the distance between the location of the height inside the triangle on the longest side, and the other shorter side. Or we multiply the sum of the two shorter [sides] by their difference, and we divide the result by the longest and subtract what results from the longest. Half of the remaining is the distance between the location of the height inside the triangle beside the shortest side next to the longest side.

Another goal is to find the measure of the height. We multiply the distance between the location of the height from one end of the base

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 في نفسه ونقص الكاقل عن مربع الضلع المتصل بذلك الطرف وناخذ
 جذر الباقي وهو العمود **مثال** لاستخراج العمود والمساواة ولما كان خط
 ب بعد موقع العمود الكاقل عن العمل الاول 4 يكون مربعه 16
 نقصناه عن مربع ا ب وهو 100 بقي 84 جذر 84 وهو مقدار العمود ضربناه
 في 10 نصف قاعته المثلث الاول حصل 84 وهو المساواة موافقا
 لما سبق **طريق اخر** ان كانت احدى زوايا المثلث معلومة فنضرب جيبها
 في احد الضلعين المحيطين بتلك الزاوية ونقسم الكاقل على ستين ليخرج
 العمود الواقع على الضلع الاخر ولو فعلنا بحيث تمامه هكذا يحصل بعد
 موقع العمود عن هذه الزاوية وسنورد معنى الجيب وجدوله **مثال**
 كان زاوية ا ب 7 من المثلث المذكور على ما ينبغي **في ربط جيب 7**
 ضربناه في ضلع ا ب وهو عشرة وقسمنا الكاقل على ستين خرجت
 من التمام ثمانية وهي العمود على ضلع 7 ومنها معرفة زوايا
 المثلث اذا كان الاضلاع معلومة يحصل العمود كما ذكرنا ثم نضرب العمود
 في ستين ونقسم الكاقل على كل واحد من الضلعين المتصلين برأس
 العمود ليخرج جيب الزاوية التي يحيط بها القاعد وذلك الضلع المقسوم عليه
 بقومته في الكدول ليحصل مقدار كل واحد من الزاويتين فان وقع العمود
 داخل المثلث فنقص مجموعهما عن ثمانية وثمانين بقيت الزاوية الباقية وان
 وقع خارجا عنه باحد النوازل بينهما وهو الزاوية الباقية **مثال** ضربنا العمود
 الكاقل وهو 8 في ستين حصل 84 قسمناه على كل واحد من ضلعي ا ب

by itself, and we subtract the result from the square of the side that is connected to that end. The square root of the result is the height.

Example. Extracting the height and the area. Consider BD , the line segment between the foot of the height obtained earlier and the vertex, which has length 6. Its square is 36. We subtract it from the square of AB which is 100, the remainder is 64. Its square root is 8, which is the length of the height. We multiply it by $\frac{10}{2}$ which is half the base of the first triangle, and we get 84 as the area agreeing with the earlier [computations].

Another Way. If one of the angles of a triangle is known, we multiply its sine by one of the sides enclosing that angle. We divide the result by sixty to get the height on the other side. And if we do the same with the sine of its complementary angle, then we get the distance between the height and that angle. We will provide the definition of sine and its table.

Example. The angle ABC from the previous triangle is $53;07 : 49^{25}$ and its sine is $48;00 : 00^{26}$ we multiply it by the side AB which is ten and divide the result by sixty. The result of this division is eight, which is the height on the side BC .

Another goal is to find the angles of a triangle whose sides are known. We obtain the height as mentioned, then multiply it by 60 and divide the result by each one of the sides connected to the vertex from which the height is drawn. The result would be the sine of the angle enclosed by the base and the side that we divided by. We look up the arc in the table to find the measurement of each of the two angles.²⁷ If the height is inside the triangle, we subtract the sum of the two angles from 180, so that the remaining angle is obtained. And if the height is outside [the triangle], we take the difference between the two angles. The result is then the remaining angle.

Example. We multiply the height obtained, which is 8, by 60 and we get 480. We divide it by each of the sides AB

²⁵ We separate sexagesimal digits by colons, and we use a semicolon to separate the whole part from the fractional part.

²⁶ The angle $53;07 : 49$ in radians is 0.9273 and $\sin(0.9273) = 0.8000$ which is $48;00 : 00$ in sexagesimal numbers.

²⁷ Al-Kashī is using the function inverse sine.

٦ من المثلثين المسبوقين خرج من الاول ٤ ٤ ٤ ومن الثاني ٤ ٤ ٤
 تسمى هاتين الجهتين الخارج من الاول ٤ ٤ ٤ وذلك مقدار زاوية ب من
 المثلث الاول وتماها من المثلث الثاني الى قائمتين وخرج من تقويس الثاني
 ٤ ٤ ٤ وهو مقدار زاوية ٦ من المثلثين ومنهما ما كان ضلع
 وزاويان معلوما والباقي مجهولا فنقص مجموع الزاويتين عن مائة وثلاثين
 يبقى الزاوية الباقية ثم نضرب الضلع المعلوم في جيب كل واحد من
 الزاويتين اللتين على طرفيه ونقسم الكا صل على جيب الزاوية التي يتوزع
 الضلع المعلوم فما خرج فهو الضلع الموتر للزاوية التي ضربنا الضلع المعلوم
 في جيبه ومنهما ما كان منه ضلعان وزاوية بينهما معلوما والباقي
 مجهولا نضرب احد الضلعين في جيب الزاوية تارة وفي جيب تمامها اخرى
 منخطا وننقص الكا صل الباقي عن الضلع الاخر ان كانت الزاوية حادة
 ونزيد عليه ان كانت منفرجة فما بلغ تربعه ونزيد عليه مربع الكا صل الاول
 وماخذ جذر المجموع فهو الضلع الباقي وان كانت الزاوية قائمة فمجموع مربعي
 الضلعين يكون مربع الضلع الباقي والحمد لله بقولنا منخطا ان حسب الاجزاء
 دقايق والدقايق ثوان وقد يطلق ذلك عند الاحتياج
 بقسمة الكا صل على سيني **مثاله** فرضنا ان من المثلث الاول ا ب ٦
 وزاوية ب معلوما والباقي مجهولا ضربنا ضلع ا ب وهو عشرة تارة في جيب
 زاوية ب الذي كان ٤ منخطا حصل ٤ وضربنا ما اخرنا في جيب تمام تلك
 الزاوية التي هو ٤ منخطا حصل ٤ ولما كانت الزاوية معلومة حادة نقصناه

كذلك

and AC from the previous two triangles [from the figures], so we get $48;00 : 00$ from the first one and $28;14 : 08$ from the second. We look up the arc in the table and we get $53;07 : 49$ from the first one, which is the measure of the angle B from the first triangle and its supplement into two right angles from the second triangle. By looking up the arc of the second [value], we get $28;04 : 22$ which is the measurement of the angle C in both triangles.

One case is when one side and two angles are known and the rest are unknown. We subtract the sum of the two angles from one hundred and eighty to get the remaining angle, then we multiply the known side by the sine of each of the two angles on its sides and divide the results by the sine of the angle opposite to the known side.²⁸ What results is the side opposite to the angle which we multiplied the known side by its sine.

Another case is when two sides and the angle between them are known and the rest are unknown. We multiply one of the sides by the sine of the [known] angle one time and by the sine of its complement the other time converted²⁹ and we subtract the second result from the other side if the angle is acute and add it if the angle is obtuse. We then square the result and add to it the square of the first result.³⁰ We take the square root of the sum to get the remaining side. If the [known] angle is a right angle, then the sum of the squares of the two sides equals the square of the remaining one.³¹ And what we mean by converted is to calculate integers as minutes, minutes as seconds, etc., which can be obtained by dividing the result by 60 when needed.

Example. We assume in the first triangle that AB , BC , and the angle B are known, and the rest are unknown. We multiply the side AB which is ten one time by the sine of B which is 48 converted [$48/60 = 0.8$] to get 8, and we multiply it [AB] another time with the sine of that angle's complement which is 36 reduced [$36/60 = 0.6$] to get 6. And since the angle is acute, we subtract it

²⁸ Al-Kāshī is using the law of sines.

²⁹ The sine of an angle converted is actually the sine of that angle divided by sixty. This is due to the fact that al-Kāshī uses a sine once elevated as per his notation in [9]. In other words, his sine is 60 times our modern sine.

³⁰ The formula is for a triangle with sides a, b, c . The sides a, b and the angle θ between them are known. Then, $c^2 = (b - a \sin(90^\circ - \theta))^2 + (a \sin(\theta))^2$. That is $c^2 = (b - a \cos(\theta))^2 + (a \sin(\theta))^2$. Expanding this last formula, and combining terms, we get the law of cosines $a^2 + b^2 - 2ab \cos(\theta) = c^2$.

³¹ This is the Pythagorean Theorem, that can be easily obtained from the law of cosines above when the angle θ is a right angle.

عن ضلع Γ وهو $\mathbf{21}$ بقي $\mathbf{15}$ مربعه $\mathbf{225}$ ومربع الحاصل الاول $\mathbf{4}$
 مجموع المربعين $\mathbf{289}$ جذره $\mathbf{17}$ وهو الضلع الباقى ومنها ما كان
 منه ضلعان وزاوية غير ما كان بينهما معلوما والباقي مجهولاً ضرب جيب
 الزاوية المعلومة في الضلع الذي يحيط مع الضلع المجهول بها ونقسم
 الحاصل على الضلع الذي يوترها فما خرج فهو جيب زاوية يوترها الضلع
 الاخر اعني الضلع المط فيه نقسمه ونزيد على الزاوية المعلومة ونقص
 المجموع عن مائة وثمانين يتبع الزاوية التي يحيط بها الضلعان المعلومان $\mathbf{23}$
 جيبه في احد الضلعين ونقسم الحاصل على جيب زاوية يوترها ذلك الضلع
 فما خرج فهو الضلع الباقى $\mathbf{24}$ ضربنا جيب زاوية $\mathbf{2}$ وهو $\mathbf{2}$ في ضلع
 $\mathbf{17}$ وهو $\mathbf{34}$ حصل $\mathbf{2}$ قسمناه على ضلع $\mathbf{17}$ وهو $\mathbf{17}$ خرج من القسمة
 جيب زاوية $\mathbf{7}$ $\mathbf{25}$ قدر قوس $\mathbf{7}$ زدناه على زاوية $\mathbf{2}$ الذي
 كان في $\mathbf{26}$ من المثلث الاول بلغ $\mathbf{23}$ فانقصناه عن $\mathbf{27}$ بقي $\mathbf{4}$ مرط
 وهو زاوية آجيبه $\mathbf{27}$ ضربنا $\mathbf{27}$ في ضلع $\mathbf{17}$ وهو $\mathbf{459}$ حصل $\mathbf{27}$ نزل
 قسمناه على جيب زاوية $\mathbf{7}$ خرج من القسمة $\mathbf{21}$ وهو ضلع $\mathbf{2}$ المطلوب
 ومنها ما كان الزاوية معلومة والا ضلع غير معلومة فلا مخلص
 سوى فرض احد الاضلاع مقداراً وليكن واحد ثم ننقسم على جيب زاوية
 يوترها الضلع المروض واحد جيب كل واحد من الزاويتين الباقيتين
 نحج من القسمة مقدار الضلع الذي يوتر الزاوية المتسوية جيبها ومنها
 العمود الخارج عن مركز المثلث اما بعمل اليد بان نصف زاويتي

from the the side BC which is 21, the remainder is 15, and its square is 225. The square of the first result is 64, and the sum of the two squares is 289. Its square root is 17, which is the remaining side.

Another case is when two sides and an angle other than the one between them are known, and the rest are unknown. We multiply the sine of the known angle by the known side among the two sides that enclose it. We divide the result by the side opposite to the angle, the result is the sine of the angle opposite to the other side, i.e., the side we are looking for.³² We obtain the arcsine of it and add it to the known angle, then we subtract the sum from 180. What is obtained is the angle enclosed by the two known sides. We multiply its sine by one of the two sides and divide the result by the sine of the angle opposite to that side, the result is the remaining side.

Example. We multiply the sine of the angle B which is 48 by the side AB which is 10 and we get 8;00. We divide it by the side AC which is 17. The result of this division is the sine of the angle C , 28;14 : 08, with arcsine 28;04 : 22 that we add to the angle B which is 53;07 : 49 from the first triangle. So, we get 81;12 : 11, which we subtract from 180 and get 98;47 : 49 as a remainder, which is the angle A . The sine of this angle is 59;17 : 39. We multiply it by the side AB which is 10 to get 9;52 : 56 : 30. We divide it by the sine of C . The result of the division is 21 which is the side BC that we wanted.

Another case is when the angles are known and the sides are unknown. The only way to solve this is to assume the length of one of the sides to be some number, let it be 1. Then, we take the sine of the angle opposite to that side and we divide the sine of each one of the other angles by that sine. The result of the division is the length of the side that is opposite to the angle whose sine is the divisor.

Another goal is to obtain the perpendicular coming out of the center of the triangle. This can be obtained by drawing: we bisect two angles

³² It is written here “looking for” but it seems to be “multiplied by” instead.

منه بطين فملتقا هما هو مركزه يخرج منه عمودا على احد الاضلاع فهو العمود
 واما بحساب فنضرب احد الضلعين في الاخر ونقسم الحاصل على
 مجموع الاضلاع الثلاثة فما خرج بضرب في جيب الزاوية التي يحيط بها المضرب
 ونقسم الحاصل على الستين فما خرج فهو العمود الخارج عن مركز المثلث
 على كل واحد من اضلاعه **مثاله** في المثلث المسبوق ضربنا العشرة في
 ٢١ حصل ٢١٠ قسمناه على مجموع الاضلاع وهو ٢٧ فخرج من القسمة **٧**
 ضربناه في جيب زاوية ا ب ٧ الذي كان ٦٦ حصل ٢١٠ قسمناه على
 ستين خرجت من القسمة ثلثه ونصف وهو العمود الخارج عن مركز المثلث
 على الاضلاع ضربناه في نصف مجموع الاضلاع الذي هو ٢٧ حصل ٢٧
 وهو المساحة كما سبق بعينه واستخرج هذا العمود بهذا الطريق مما
 استنبطنا **الفصل الثالث** في مساحة المثلث
 المتساوي الاضلاع كتحصيل واستخراج ابعاده بعضها عن بعض
 اما المساحة فليكن في الاضلاع من المثلث طرق اخرى غير مارة
 بالاول ان نأخذ مال مال نصف احد اضلاعه ونضربه في الثلثة
 دائما ونأخذ جذر الحاصل فهو المساحة **الثاني** نأخذ جذر ثلث
 مال مال العمود يحصل المساحة **الثالث** نضرب ربع احد اضلاع
 في **١٠** حاصل المساحة **الرابع** نضرب نصف ثلث
 جميع الاضلاع في مكعب ضلع واحد او نقسم ضلعا واحدا على خمسة و
 ونضرب الخارج في مكعب ضلع واحد ونأخذ جذر الحاصل فهو المساحة

١٠ ثلثه وعشر مربع ضلع المثلث

in the triangle using two lines. The point of intersection of these two lines is the center of the triangle. We draw from that point a perpendicular [line segment] on one of the sides, and that is the required.

As by calculations, we multiply one of the sides by the other, and we divide the result by the sum of the three sides. Then we multiply the result by the sine of the angle enclosed by the two multiplied sides and we divide the result by 60. The result is the height coming out of the center of the triangle on each one of the sides.³³

Example. In the previous triangle, we multiply ten by 21 to get 210. We divide it by the sum of the sides, which is 48. The result of the division is $4;22:30$. We multiply it by the sine of the angle ABC which is $48;00$ to get 215. We divide it by sixty. The result of the division is three and a half which is the height coming out of the center on the sides. We multiply it by half the sum of the sides which is 24 to get 84, which is the area as shown before. And obtaining this height this way is from our investigation.

Third Section. Specifically on the area of an equilateral triangle, and obtaining its dimensions.

There are ways to obtain the area of an equilateral triangle other than the ones mentioned before. **The first** one is to take the fourth power of half of one of its sides and multiply it always by 3. The square root of the result is the area. **The second** is to take the square root of a third of the fourth power of the height to get the area. **The third** is to multiply the square of one of its sides by $25:58:50:44:37$ fifth. The result is the area. **The fourth** way is to multiply half of the eighth of the sum of all sides by the cube of one side, or we divide one side by five and a third and multiply the result by the cube of one side, then we take the root of the result and that is the area.³⁴

³³ The formula here is for a triangle with sides a, b, c and angle α between the sides a and b , the inradius is $\frac{ab \sin(\alpha)}{60(a+b+c)}$. Recall that al-Kāshī is dividing by 60 because his sine is 60 times our modern sine. In modern mathematics, the inradius is $\frac{ab \sin(\alpha)}{(a+b+c)}$.

³⁴ There are two ways of computing the area of an equilateral triangle with sides a . In either case, we obtain the area as $\frac{a^2\sqrt{3}}{4}$. In the right margin there is an approximation of the area: "It is a third and a tenth of the square of the side of the triangle." The approximation of the area is $\frac{13a^2}{30}$.

60

أما استخراج الأبعاد بعضها عن بعض إذا أخذنا جذر ثلثة أرباع مربع
 ضلع واحد فهو العمود وثلث العمود هو العمود الخارج من مركز المثلث
 أي نصف قطر دائرة وقعت فيه تماس جميع أضلاع أضلاعه وإذا
 زدنا على مربع العمود الخارج من الزاوية على القاعدة ثلث المربع فمأخذ
 المبلغ يحصل مقدار ضلع منه وإذا أضربنا ضلعه في **ناز ماسط مد** فإنه
 يحصل العمود وإذا أخذنا ثلث مربع ضلع واحد فمأخذ جذره يحصل نصف قطر
 دائرة احاطت به وتماس زواياها وإذا أخذنا نصف سدس مربع ضلع
 واحد ويحصل جذره فهو العمود الخارج عن مركزه إلى منتصف ضلعه ويكون
 في هذه المثلث مركز الدائرة الداخلة المماس لأضلاعه والخارجة
 المماس لزاوياها واحدا بخلاف مختلف الأضلاع **الباب**
الثاني في مساحة ذوات الأربعة الأضلاع وما يتعلق بها
 ويشمل على خمسة فصول **الفصل الأول** في تعريفات
 ذوات الأربعة أضلاع سطح محيط به أربعة خطوط مستقيمة وهو ينحصر إلى تسعة
 الأضلاع وممكنها ومتساوي الزوايا وممكنها فيصير أربعة أنواع **الأول**
 متساوي الأضلاع والزوايا يسمى مربعا **الثاني** متساوي الزوايا ومختلف
 الأضلاع يسمى مستطيلا وبها تشارك في تساوي القطرين أي الخطين
 الواصلين بين كل زاويتين المتقابلتين **الثالث** متساوي الأضلاع
 مختلف الزوايا يسمى معيناً وهو مع **الأول** يشترك في تقاطع القطرين
 على قوائم والثلثة في توازي الأضلاع **الرابع** ممكن الأضلاع

As for obtaining some dimensions from others: If we take the square root of three quarters of the square of one side, then we get the height. A third of the height is the height coming out of the center of the triangle. I mean, it is the radius of the inner circle that is tangent to all of its sides at their midpoints. If we add the square of the height to its third and take the square root of the result, then we get the length of one side. And if we multiply the side by 51 : 57 : 41 : 29 : 14 fifth, we get the height. If we take a third of the square of one side and take the square root of that, then we get the radius of the outer circle that contains its vertices. And if we take half of the sixth of the square of one side and take the square root of that, then we get the height coming out of the triangle's center to the middle of its side. In this triangle [equilateral], the center of the inner circle that is tangent to its sides is the same as the center of the outer circle touching its vertices, unlike triangles with different sides.

Second Chapter

On the area of quadrilaterals and related matters

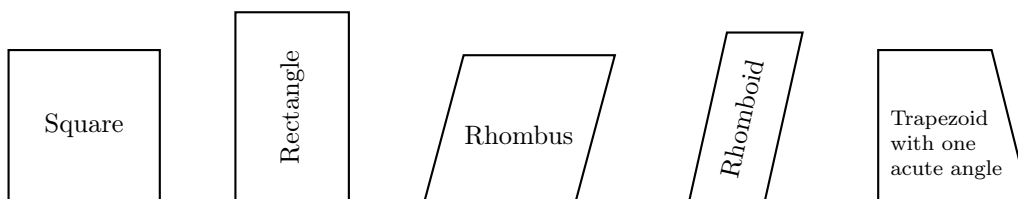
First Section. Definitions of quadrilaterals. It is a surface enclosed by four straight lines and is classified into four types with equal and different sides and equal and different angles so that they become four: **First:** When the sides are equal and the angles are equal, it is called a square. **Second:** When the angles are equal and the sides are different, it is called a rectangle. The two [square and rectangle] share the equality of their diagonals, meaning the two lines connecting opposite vertices. **Third:** When the sides are equal and the angles are different, it is called a rhombus. A rhombus shares a property with the first [square] in which both have diagonals intersecting perpendicularly. All three share the property that opposite sides are parallel. **Fourth:** Different sides

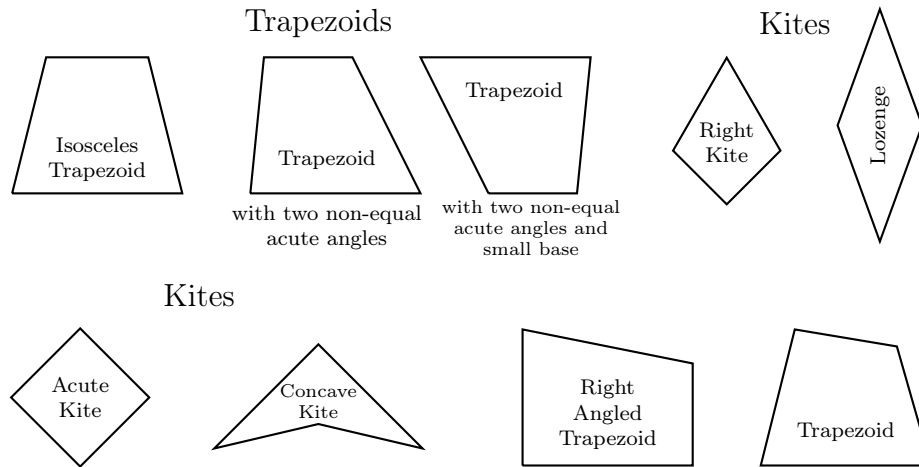
والزوايا وهو ما يكون كل ضلعين متقابلين منه متوازيين متساويين لكن
غير متساويين للآخرين سمى بشبيه المعين وهو مشترك للثلاثة الأول
في توازي الاضلاع واما ان يكون ضلعان منه متوازيين والآخران غير
متوازيين سمى بذى الزنقة وذى الجناح وهو ثلثة انواع **أ** ذوزنقة
واصح وهو ما كان احد الضلعين الغير المتوازيين عمودا على المتوازيين
ب ذوزنقتين متساويتين وهو ما يتساوى فيه الضلعان الغير
المتوازيان **ج** مختلف الزنقتين وهو ما كان فيه الضلعان
الغير المتوازيان غير متساويين ولا يكون احدهما عمودا على المتوازيين
وقد يكون هذا الاختلاف في الحكمة ايضا واما ان يكون فيه ضلعان
متساويان متساويين وكذا الاخران والاولان يحالان الاخرين
ووقع تقاطع قطاره في داخله سمى بذى اليمينين ويكون فيه
لامحالة زاويتان متقابلتان متساويتين فقط اما قائمتين
فيسميه البنانون باللوزة واما منفرجتين ونسميه النجارون بجودانه
واما حادتين ونسميه الباطية ويتقاطع قطاره هذه الثلثة على قوائم
كالمربع والمعين وتقام ذى اليمينين الى المعين نسبة بدن رجلين
وما لم يكن على هذه الاشكال سمى منحفا وهو ما ان يكون احد
زواياه قائمة سمى منحفا قائم الزاوية والافغير قائم الزاوية وهذه

صورها



and angles. It is either that each pair of opposite sides are parallel and equal, but not equal to the other two. It is called rhomboid and it shares a property with the first three in which they all have parallel sides. Or it has two parallel sides but the other two are not, then it is called a trapezoid [that is not a parallelogram] and it is of three types. **(A)** A trapezoid with one acute angle is when one of the nonparallel sides is perpendicular to the other parallel pair. **(B)** A trapezoid with two equal acute angles is one where non-parallel sides are equal. **(C)** A trapezoid with two unequal acute angles. In this case, non-parallel sides are not equal and none of them is perpendicular to any of the parallel sides. And this difference may also be in the direction as well. In the case where each pair of adjacent sides are equal to each other, one pair is different from the other pair and the intersection of the two diagonals is inside, and it is called a kite. A kite must have two equal opposite angles only. Either they are right angles, then the builders call it a right kite, or they are obtuse and carpenters call it a lozenge [obtuse kite] or they are acute and we call it an acute kite. The diagonals of these three [types] intersect perpendicularly like a square and a rhombus. A kite that can be completed to be a rhombus is called a concave kite. Any shape other than the ones described above is called a trapezoid. If one of its angles is right then it is called a right trapezoid. Otherwise, it is a non-right angled trapezoid. Here are their pictures.





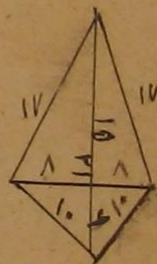
Second Section. On the area of a square and a rectangle and obtaining some dimensions from others. The area is obtained by multiplying the length by the width, i.e., one of the sides by its adjacent side. **Another way** is we multiply ~~the length~~ by one of the diagonals by the height that comes out of one of the remaining vertices. In a square, that is half of the diagonal. As for obtaining some dimensions from others, we take the square root of the sum of the squares of adjacent sides which is the diagonal.³⁵ Then the square of the diagonal in that square is two times the square of its side. If we multiply the side of the square by 1; 24 : 51 : 10 : 7 : 46 fifth³⁶, we obtain its diagonal. If we divide the diagonal by that [number] or multiply it by half [of the number], i.e., by 0; 42 : 25 : 35 : 3 : 58 fifth, we obtain its side. The extraction of the height that comes out of a vertex of a rectangle on its diagonal is like extracting the height of a triangle.

Third Section. On the area of a rhombus and a kite and obtaining dimensions from one another. To obtain the area, we multiply one of the diagonals by half of the other. This is the same for the square. To obtain the area of a rhombus in particular, we subtract

³⁵ This is the Pythagorean Theorem.

³⁶ This is $\sqrt{2}$ in base sixty.

مربع الفضل بين نضع القطرين عن مربع احد اضلاعه فيكون الباقي
 مساحة **مسألة** معين يكون كل واحد من اضلاعه عشرة وقطره
 الاطول ستة عشر والاقصى عشرة فاذا ضربنا ستة في ستة عشر
 حصلت المساحة وهي ستة وتسعون واذا اخذنا تفاضل نضع
 القطرين وهو اثنان ونقصنا مربعه وهو اربعة عن مربع احد اضلاعه
 وهو مائة بقى ايضا ستة وتسعون ويكتفى بمساحة ذوات اليمينين
 ان ننقص مجموع مربعي التفاضل بين نصف قطر الذي بنصف
 بالاخر وبين كل واحد من قسمي الاخر اللذين ينفصلان بالقطر
 الاول عن مجموع مربعي الضلعين المختلفين وبنصف الباقي فهو
 المساحة **مسألة** في ذى يمينين يكون كل واحد من ضلعيه
 الاقصرين عشرة ومن الاطولين سبعة عشر وقطره الاقصى
 ستة عشر والاطول لهذا وعشرين فاذا ضربنا الثمانية في **٢١**
 حصل المساحة **١٤٨** فاذا اخذنا فضل نصف قطر الاقصى على
 كل واحد من قسمي الاطول كان احدهما **٢** والاخر **٥** كما ظهر في
 المثلث الاول في الفصل الثامن من ابواب الاول وسيظهر هذا في
 استخراج الابعاد فجمعنا مربعيهما كان **٥٣** نقصناه عن مجموع
 مربعي الضلعين المختلفين وهو **٣٨٩** بقى **٣٣٦** نصفناه صار
١٦٨ وهو المساحة موافقا بحساب الاول وما كانت زاويتا
 منه قائمتين يحصل مساحة ضرب احد الضلعين المختلفين في

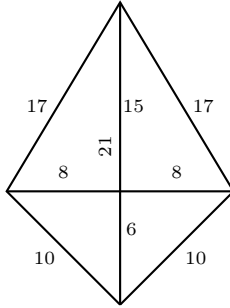


the square of the difference between half of the diagonals from the square of one of its sides. The remainder will be the area.³⁷

Example. Consider a rhombus each of whose sides is ten, its longer diagonal is sixteen, and the shorter one is twelve. If we multiply six by sixteen we get the area, which is ninety-six. If we take the difference between half of the diagonals, which is two [$8 - 6 = 2$], and subtract its square, which is four, from the square of one of its sides, which is one hundred, the remainder is ninety-six as well.

To find the area of a kite in particular, we subtract the sum of the squares of the differences between half of its diagonal that is bisected by the other and each one of the two parts that the first diagonal separates from the sum of the squares of the two different sides.³⁸ Half of what is left is the area.

Example. In a kite, let each of the two shorter sides be ten and the longer sides be seventeen. Its shorter diagonal is sixteen and the longer one is twenty-one. If we multiply eight by 21, we get the area to be 168. If we take the difference between half of the shorter diagonal and each one of the two parts of the longer diagonal, one of them will be 2 and the other one will be 7, as it appears in the first triangle in the second section from the first chapter. It will also appear in extracting the dimensions. We add their squares and get 53.



We subtract it from the sum of the squares of the two different sides, which is 389, and 336 is left. We halve it and it becomes 168 which is the area according to the first computation.

If two of its angles are right angles, then the area is obtained by multiplying one of the unequal sides by

³⁷ For a rhombus with a side a and diagonals d_1 and d_2 , the area is $a^2 - \left(\frac{1}{2}d_1 - \frac{1}{2}d_2\right)^2 = \frac{1}{2}d_1d_2$.

³⁸ Consider a kite with the different sides being a and b , and a diagonal d that is halved by the other diagonal. Assume that diagonal d cuts the other diagonal into two segments d_1 and d_2 . Then, the formula for the area of the kite is $\frac{1}{2} \left(a^2 + b^2 - \left[\left(\frac{d}{2} - d_1 \right)^2 + \left(\frac{d}{2} - d_2 \right)^2 \right] \right)$. If we expand and use the Pythagorean Theorem, we obtain the area as being one-half the product of the diagonals of the kite.

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الاخر واما استخراج ابعادها عن بعض نظرب جيب نصف احدى
 المعين في احد الضلعين المحيطين بها ونقسمها حاصل على ستين فما خرج فهو
 نصف القطر التي يوتر تلك الزاوية وكذا الحكم في دوات اليمينين
 اذا عملنا على زاويتي المختلفتين لا المتساويتين ذلك العمل ضعف
 خارج القسم هو القطر الموتر لتلك الزاوية اعني الواصل بين الزاويتين
 المتساويتين وان اردنا استخراج القطر الواصل بين الزاويتين
 المختلفتين نأخذ نصف تمام كل واحد من الزاويتين المختلفتين وننظر
 جيبه في الضلع المحيط بتلك الزاوية ونقسمها حاصل على ستين لينخرج
 كل واحد من قسم القطر المذكور مجموعها ليحصل القطر وان كان احد
 قطري المعين معلوما فنقص مربع نصف عن مربع احد اضلاعه
 يبقى مربع نصف قطره الاخر وان كان القطر الواصل بالزاويتين
 المتساويتين لذوات اليمينين معلوما فنقص مربع نصفه عن كل
 واحد من مربعي الضلعين المختلفتين ليبقى كل واحد من مربعي قسمي
 قطره الاخر **مثاله** في دنى اليمينين المذكور كان نصف قطره
 الاصغر ٦ مربعه **٤٣** فنقصناه ثمانية عن مربع ضلعه الاصغر وهو
١٥٥ بقي **٣٤** جذره **٥** وهو اصغر قسم قطره الاطول فنقصناه
 اخرى عن مربع ضلعه الاطول وهو **٣٨٩** بقي **٢٢٩** جذره **١٥** وهو
 اطول قسميه وان كان قطره الواصل بالزاويتين المختلفتين معلوما
 فنعينه بذلك القطر مثلثين فيحصل نصف قطره الاخر كما حصلنا

the other. As for obtaining dimensions from others, we multiply the sine of half of one of the angles of the rhombus by one of the sides enclosing it. And we divide the result by sixty. What comes out is half the diagonal opposite to that angle.³⁹ It is the same rule for the kite if we apply the same process on one of the two different angles, not the equal ones. The double of the outcome of the division will be the diagonal opposite to that angle. I mean the one connecting the two equal angles. If we want to extract the diagonal connecting the two different angles, we take half of the complementary angle of each one of the different angles and we multiply its sine by the enclosing side of that angle. We divide the outcome by sixty to obtain each one of the two parts of the mentioned diagonal. We add them together to get the diagonal. If one of the rhombus' diagonals is known, then we subtract the square of its half from the square of one of the sides. The square of half of its other diagonal remains. If the diagonal connecting the two equal angles to the kite is known, then we subtract the square of its half from each one of the two different sides' squares to get the square of each part of its other diagonal.

Example. In the mentioned kite, half of the shorter diagonal is 8. Its square is 64. We subtract it once from the square of the shorter side which is 100 and 36 remains. Its square root is 6 and it is the smaller part of its longer diagonal. Then, we subtract it [64] from the square of the larger side which is 289 and 225 remains. Its square root is 15 which is the longer part of the diagonal. If its diagonal connecting the two different angles is known, then that diagonal makes two triangles. So we obtain half of the other diagonal like we did

³⁹ This is a straightforward use of the definition of sine in a right triangle.

عمود المثلث **العضد الرابع** في مساحة الشبيه بالمعين
وذوات الزنقة واستخرج الابعاد بعضها عن بعض امتسا
المساحة فيحصل ضرب العمود الخارج من احدى زواياه على احد المتوازيين
في نصف مجموع المتوازيين اللذين وقع العمود عليها ونشترك
فيه المعين ايضا وامسا معرفة العمود اما بعمل اليد فعمل قياس
ما حره المثلث وامسا بالكتاب في ذى الزنقتين المتساويتين
فناخذ جذر التفاضل بين مربع نصف متااصل المتوازيين ومربع احد
الاخرين وفي ذى زنقة واحد هو اقصر الضلعين اللذين ليسا
بمتوازيين وهو مساو لجذر التفاضل بين مربع الضلع الاعظم
من الضلعين المذكورين ومربع تفاضل المتوازيين وفي ذى
الزنقتين المختلفتين اذا كانت الزاوية التي يحيط بها اطول
المتوازيين واقصر الاخرين حادة اعني يكون ضاهاه في جهة
واحد يحصل العمود كما حصل في المثلث اى نسقط اقصر المتوازيين
ومثل الاطول ليصير كمثلث ونجعل الباقى قاعدة المثلث
ويحصل العمود بوجه من الوجوه المذكور في المثلث وهذا الطريق
شامل لجميع انواع ذوات الزنقة وفيما لا يكون فيه في جهة واحد
وفي الشبيه بالمعين ان كانت احدى زواياه معلومة ف ضرب
جيب تلك الزاوية في اقصر الضلعين المحيطين بها من خط
فما حصل فهو العمود كما ذكرنا في المثلث ولو ضرب جيب الزاوية

in the height of a triangle.

Fourth Section. On the area of a rhomboid and a trapezoid [that is not a parallelogram], and obtaining dimensions from each other. To obtain the area we multiply the height coming out of one of the vertices on one of the two parallel sides by half the sum of the two parallel sides that the perpendicular is located on. The rhombus shares this property too. As for knowing the perpendicular, it is done either by hand [drawing], just like what we did before in a triangle, or by calculation. In a trapezoid with two equal angles, we take the square root of the difference between the square of half of the difference between the parallel sides and the square of one of the other [sides]. In a trapezoid with one acute angle, the height is the shorter of the two nonparallel sides which is equal to the square root of the difference between the square of the greater of the two sides mentioned and the square of the difference between the two parallels. In a trapezoid with two different acute angles, if the angle that the longer of the two parallels and the shorter of the two other sides enclose is acute, meaning its [the trapezoid's] two wings are in one direction, then we can obtain the height as we did in the triangle, i.e., by projecting the shorter of the two parallel sides on the longer side obtaining a triangle.⁴⁰ We make the remaining part a base to the triangle and we obtain the height using any of the ways mentioned in the triangle [section]. This method is for all types of trapezoids even if they are not in one direction. In a rhomboid, if one of its angles is known, then we multiply its sine by the shorter of the of the two sides enclosing it reduced [by a factor of 60]. What results is the height, as we mentioned in the triangle [section]. If we multiply the sine of the angle

⁴⁰ See ([6], P. 230) for a figure explaining this case.

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الشبه بالمعنى في احد الضلعين المحيطين من خطا ليحصل العمود الواقع
على الضلع الاخر وان لم يكن معلومة فلا مخلص سوى عمل اليد
الفصل الخامس في مساحة ذي الرجلين والمنحرف

نصل بين زاويتين متقابلتين منه خطا مستقيما ليصير مثلثين
ونقسمهما ونجمع احوالين فهو المراد ونشترك فيه جميع ذوات
الاربعة الاضلاع وما يخص بذى رجلين ان نصل بين زاويتي
رجليه خطا مستقيما ونقسم المثلث الاصفى الحادث وننقصه
عن مساحة المثلث الاعظم فما بقى فهو المراد او نضرب نصف
ذلك الخط في الخط الواصل بين زاويتي الباقيتين وما قيل في
مساحة الشكل المستقيم نقشا وهو ايضا منحرف ليس بصحيح
فلا نورد. واما استحتاج ابعاده ان كان بعض زواياه معلوما
فيحصل بعض الابعاد على قياس المثلث بعد تقسيمه مثلثين
والا فيحصل الاعمدة بعمل اليد على ما سبق **الباب**

الثالث في مساحة ذي الاضلاع الكثيرة وما يتعلق
وهو مشتمل على خمسة فصول **الفصل الاول** في تعيين
ذو الاضلاع الكثيرة سطحه كخط به خطوط مستقيمة اكثر من اربعة كالمخمس
والسدس والسبع والمثلث وما بعدهما وهو اما متساوي
الاضلاع والزوايا واما مختلف فهما واما احدهما متساوية
الاخرى مختلفه وقد يمكن ان يرسم في الاول اية تامة جميع اضلاعه

of a rhomboid by one of the enclosing sides reduced [by a factor of sixty] to get the height located on the other side, and if the angle is unknown, then there is no way to solve it except by hand [drawing].

Fifth Section. On the area of concave kites and obtuse trapezoids. We connect two opposite vertices with a straight line so that two triangles are formed. We measure them and add the outcomes, which is what we want. All quadrilaterals share this.⁴¹ For the concave kite, we connect the vertices of the two legs with a straight line and we calculate the area of the smaller triangle that is formed. We subtract it from the area of the greater triangle. What remains is what we want. Or we multiply half of that line by the line connecting its two remaining vertices. What is said on the area of the figure that is called a trapezium, which is also a trapezoid, is not valid. So, we do not include it [here]. As for obtaining its dimensions, if some of its angles are known we obtain some of its dimensions based on the measurement of the triangle after triangulating it into two triangles. Otherwise we obtain the heights by hand as before.

Third Chapter On the area of polygons and related matters

It contains five sections.

First Section. On definitions. A polygon is a surface surrounded by more than four straight line segments such as a pentagon, hexagon, heptagon, octagon, and so forth. A polygon can have equal sides and angles or unequal sides and unequal angles, or all sides may be equal with unequal angles or all angles may be equal with unequal sides. In the case of a regular polygon, it is possible to draw a circle that is tangent to every side.

⁴¹ This is triangulation of polygons.

This is also true in some of the other cases.

Second Section. On the area in general and obtaining dimensions from each other. Finding the area in the most general case is to triangulate the polygon, to measure the areas of all triangles, and to add all of them up.

Another method. If it is possible to draw an inner circle that is tangent to all sides, which in a regular polygon is at the midpoint, we multiply the radius of that circle by half of the sum of all sides to get the area. To obtain the radius of that circle by hand, we bisect two of the figure's angles by two intersecting lines, and the point of intersection is the center of the circle. We then draw a perpendicular from the center onto one of the sides and measure it. To obtain it by calculation, we multiply the sine of half of one of its angles by the sine of half of the complement of the other angle that is adjacent to the first angle. Then, we divide the result by the sine of half of the second angle. What results is added to the sine of the complement of half of the first angle. Then, we divide by this sum, the product of the sine of half of the first angle and the side between the two angles. What is obtained is the length of the radius of the circle in the same units used for the side.

Third Section. Specifically on regular polygons not considered above and obtaining some of the dimensions from others. To compute the area we multiply the square of one side of the pentagon by 1; 43 : 13 : 43 : 7 : 8 fifth; of the hexagon by 2; 35 : 53 : 4 : 27 : 42 fifth; of the heptagon by 3; 38 : 2 : 5 : 18 : 40 fifth; of the octagon by 4; 49 : 42 : 20 : 15 : 32 fifth; of the nonagon by

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وحده لدع تو فام والمعشر في ر مالط طوله فامه وذي
 اثني عشر ضلعاً نما موح نه الذ فامه وذي خمسة عشر ضلعاً في
ر ح ل ب ل ط فامه وذي ستة عشر ضلعاً في ك و ل م
ط فامه ليحصل مادة ذلك المضلع وهذه الاعداد
 في امثال مربع ضلع واحد و اجزائه لذلك المضلع وقد وضعنا
 بالارقام والكتابتة معاً مع اضعا في جدول اذ لو وقع
 عند نقل النسخ منه غلط ليسهل تصحيح الارتباط بعضها ببعض
 وايضاً هولنا هذه المقا دير الى الرقوم الهندية لكن ليس
 بتلك الدقة بل اخذنا الكثير كلها من مخرج واحد وهو الف
 الف ليكون على قياس صاحب المنجمين اذ يحصل
 للصحيح اعشار ثم للاعشار اعشار التي سمينها بأشانه
 الاعشار ثم للاعشاره اعشار المسما بثالث الاعشار
 وهكذا الى سادس الاعشار

ووضعنا بها
 ايضا جدول
 بالارقام
 والكتابتة
 والتصنيف
 واجداول هذه

6;10 : 54 : 18 : 16 fifth; of the decagon by 7;41 : 39 : 9 : 6 : 35 fifth, of the dodecagon by 11;11 : 46 : 8 : 55 : 34 fifth; of the pentadecagon by 17;38 : 32 : 30 : 23 : 19 fifth; and of the hexadecagon by 20;6 : 33 : 41 : 19 : 16 fifth to obtain the area of that polygon.⁴² These numbers are equivalent to the square of one side and a multiplier specific to that polygon. We put them in both numerical and written forms as rows of a table so that if there is a mistake in copying them from one place to another, it will be easy to correct due to the connection between them. Also, we converted these amounts to *Indian numerals*, but not in the same precision. Rather, we take all the fractions in the same denominator, which is a million, so that it is compatible with astronomers' [sexagesimal] arithmetic. So, an integer is divided into ten parts, then a part is divided into ten called second tenth, then third tenth, and so on until sixth tenth. We also put these numbers in a table in numerical and written form as well as their doubles. Here is the table.

⁴² Given the formula for the area of a regular polygon with n sides, these numbers are $\frac{n}{4 \tan(180/n)}$ written in sexagesimal numbers.

[illegible]

Ratio of the area of regular polygons to the square of one side in sexagesimal numbers.

Regular Polygons	Area of Polygons						Numbers in Writing						Doubles					
	Parts	Minutes	Seconds	Thirds	Fourths	Fifths	Example square of one side	Minutes	Seconds	Thirds	Fourths	Fifths	Parts	Minutes	Seconds	Thirds	Fourths	Fifths
Triangle	0	25	58	50	44	37	Zero	Twenty-five	Fifty-eight	Fifty	Fourty-Four	Thirty-seven	0	51	57	41	29	14
Pentagon	1	43	13	43	7	8	One	Forty-three	Thirteen	Forty-three	Seven	Eight	3	26	27	26	14	16
Hexagon	2	35	52	4	27	42	Two	Thirty-five	Fifty-three	Four	Twenty-seven	Forty-two	5	11	46	8	55	24
Heptagon	3	33	2	5	18	40	Three	Thirty-eight	Two	Five	Eighteen	Forty	7	6	4	10	37	20
Octagon	4	49	42	20	15	32	Four	Forty-nine	Forty-two	Twenty	Fifteen	Thirty-two	9	39	24	40	31	4
Nonagon	6	10	54	34	18	16	Six	Ten	Fifty-four	Thirty-four	Eighteen	Sixteen	12	21	49	18	36	32
Decagon	7	41	39	9	6	35	Seven	Forty-one	Thirty-nine	Nine	Six	Thirty-five	15	23	18	18	13	10
Dodecagon	1	11	46	8	55	24	Eleven	Eleven	Forty-six	Eight	Fifty-five	Twenty-four	22	23	32	17	50	48
Pentadecagon	17	38	32	30	23	19	Seventeen	Thirty-eight	Thirty-two	Thirty	Twenty-three	Nineteen	35	17	5	0	46	38
Hexadecagon	20	6	33	41	19	16	Twenty	Six	Thirty-three	Forty-one	Nineteen	Sixteen	40	13	7	22	38	32

باب

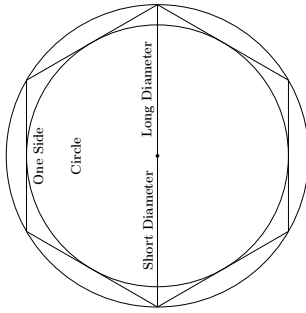
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انضام که در او ارباب بطور ارفع است
یعنی مثلث و مجسم المانیا که در
ضلع در مسدود اصل بر جدول ضرب ایجاب

Table of the ratios in Indian numerals.⁴³

Regular Polygons	Area of Polygons						In writing	Doubles						
	Parts	Tens	Its Second	Its Third	Its Fourth	Its Fifth		Its Sixth	Integers	Tens	Its Second	Its Third	Its Fourth	Its Fifth
Triangle	0	4	3	3	0	1	2	When one side of a polygon is a thousand, then the square of the side is a million, and the area of the triangle is four hundred thirty three thousand twelve in that unit and the area of the	0	8	6	0	2	4
Pentagon	1	7	2	0	4	7	7	pentagon is like the square of one side and seven hundred twenty thousand four hundred seventy seven in that unit and the area of the	3	4	4	0	9	4
Hexagon	2	5	9	8	0	7	6	hexagon is like two times the square of one side and five hundred ninety five thousand seventy six in that unit and the area of the	5	1	9	6	1	2
Heptagon	3	6	3	3	9	1	4	heptagon is like three times the square of one side and six hundred thirty three thousand six hundred fourteen in that unit and the area of the	7	2	6	7	8	8
Octagon	4	8	2	8	4	2	7	octagon is like four times the square of one side and eight hundred twenty eight thousand four hundred twenty seven in that unit and the area of the	9	6	5	6	8	4
Nonagon	6	1	8	1	8	2	5	nonagon is like six times the square of one side and one hundred eighty one thousand eight hundred twenty five in that unit and the area of the	12	3	6	3	6	0
Decagon	7	6	9	4	9	0	9	decagon is like seven times the square of one side and six hundred ninety four thousand nine hundred nine in that unit and the area of the	15	3	8	9	8	1
Dodecagon	11	1	9	9	1	5	2	dodecagon is like eleven times the square of one side and one hundred ninety nine thousand one hundred fifty three in that unit and the are of the	22	3	9	8	3	0
Pentadecagon	17	6	4	2	3	4	3	pentadecagon is like seventeen times the square of one side and six hundred forty two thousand three hundred sixty three in that unit and the area of the	35	2	8	4	6	8
Hexadecagon	20	1	0	9	3	5	8	hexadecagon is like twenty times the square of one side and one hundred nine thousand three hundred fifty eight in that unit.	40	2	1	8	7	1

⁴³ This table has been checked against the same tables in manuscripts [3] and [5] as we believe some misprints slipped in. We corrected many but caution has to be taken when reading it.



Example. We want to measure the area of a regular hexagon, all of whose sides are twenty and a half cubits. We write it as 20;30 [in sexagesimal representation]. We take its square and it becomes 7 : 0;55 minutes which we multiply by 2;35;53;4;27;42 fifth. The area is obtained as in the table:

Whole Parts		Fractional Parts					
Elevated	Cubit	Minutes	Seconds	Thirds	Fourths	Fifths	Sixths
18	11	50	29	30	0	55	30

If we assume that every side is one thousand two hundred and thirty cubits, then the result is also the same numbers but the fourth digit which is 29 is cubits. What is to the right is its elevates and the remaining is its fraction, and so on.

Example. The previous area in decimal system: We take half of the cubit with the two cubits which are the length of one side as a fraction with a denominator of ten. It becomes five which we put to the right of twenty like this:

Whole	Fraction
20	5

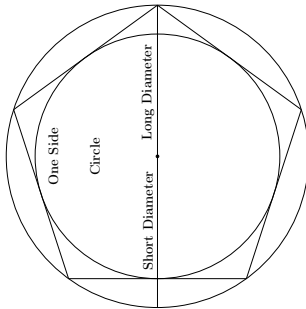
. We square it to get

Whole	Fraction
420	25

. We multiple this number

Whole	Fraction	to obtain	Whole	Fraction
2	598079		1091	841079

If we assume that every side is two hundred five cubits, the result of these numbers are also the same [themselves] but the digit four in it becomes units. I mean the integer part is 109184 and the remaining numbers are the fractional part.



Know that for every regular polygon, except the square, when its side is rational, its area is irrational.

As for extracting the dimensions, it is to obtain the radius of the circle mentioned before, i.e., the in-circle tangent at the midpoint of its sides. As by hand, when the number of sides is even, we connect the midpoints of opposite sides with straight lines. Half of that line segment

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يكون نصف قطر الدائرة المطلوبة ونما كان عدد اضلاعه فردا فنصف
 بين منصف احد اضلاعه والزاوية المتقابلة له ثم بين منصف ضلع
 اخر والزاوية المتقابلة لهذا الضلع فمن تقاطع الخطين الى منصف
 الضلع يكون نصف قطر الدائرة المذكورة والتقاطع هو مركزها
 واما بالحساب وهو ان نقسم مائة وثمانين دايما على عدد الاضلاع
 فما خرج فخذ جيبه وجيب تمامه ثم نضرب نصف درعان ضلع
 واحد في جيب تمامه تارة وفي مستين اخرين ونقسم كل واحد
 على جيبه فخرج من الاول مقدار نصف قطر الدائرة الداخلة
 ومن الثاني نصف قطر الدائرة الخارجة اعني التي تماسها واما
 الشكل وتقال لهما القطر الاطول والاقصر **نوع اخر** نقسم ساحة
 المضلع على نصف مجموع اضلاعه فما خرج فهو نصف قطر الاقصر
 ومنه استخراج الضلع فان كان نصف قطر الاطول او
 الاقصر معلوما وكان الضلع مجهولا فنضرب ما كان معلوما في
 الجيب المذكور ونقسمه كما حصل على جيب تمامه ان كان المعلوم
 نصف قطر الاقصر وعلى مستين ان كان نصف قطر الاطول
 فما خرج فنضعه ليحصل الضلع **نوع اخر** ولو كانت المساحة
 معلومة فنقسمها على ارقام ذلك المضلع ونأخذ جذر الخارج فهو
الفصل الرابع فيما يخص بالمسدس المتساوي الاضلاع
 والزاوية سلف اما المساحة فنضرب مال مال احد اضلاعه

is the radius of the desired circle. When the number of sides is odd, we connect the midpoint of one of the sides with the opposite vertex, then we connect the midpoint of another side with the vertex opposite that side. The length between the point of intersection of the two lines and the midpoint of the side is the radius of the circle mentioned. The point of intersection is the center of the circle. By calculation: We always divide one hundred and eighty by the number of sides. We take the sine of the result and the sine of its complement. We then multiply half the length of one side by the sine of its complement once, and another time by sixty. Then we divide each one by its sine. The measure of the radius of the inner circle is obtained from the first, and the radius of the exterior circle from the second, i.e., the one that touches [passes through] the vertices of the figure. These two are called the longest and shortest diameters.

Another Method. We divide the area of the polygon by half of the sum of its sides, and the result is the shortest radius.

To obtain the side if the longer radius or the shorter radius is known and the side is unknown, we multiply the known radius by the mentioned sine and divide the result by the sine of its complement if the known radius is the shorter one, and by sixty if the known radius is the longer one. The result is doubled to obtain the side.

Another Method. If the area is known then we divide it by the number of sides of this polygon and we take the [square] root of the result which is what we wanted.

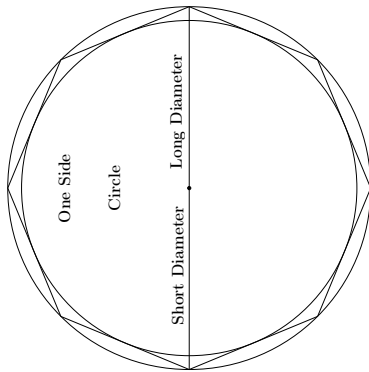
Fourth Section. Matters specific to the regular hexagon not discussed so far. As for the area, we multiply the fourth power of one of its sides

في سبعة وعشرين ونصف جذر الحاصل فهو المساد نوع آخر
 بضرب مال مال نصف قطر الدائرة الداخلة في اثني عشر فمأخذ جذر
 الحاصل فهو المط **طرق** بضرب مكعب ضلع واحد في مجده ^{ضلع} **الاحد**
 ويزيد عليه ثمن الحاصل يحصل مربع المساد ولان المسدس ^{مستوي}
 امثال مثلث متساوي الاضلاع يكون ضلعه كضلعه واما استخراج
 ابعاده فمأخذ جذر ثلث امثال مربع ضلعه يكون قطر الاقصر ويضعف
 عمود مثلث متساوي الاضلاع هو سدسه وقطره الاطول ضعف
 ضلعه **الفصل الخامس** فيما يخص بالمتمثل المتساوي الاضلاع
 والزوايا غير متساوية استخراج ابعاده اما المساد فنقص مربع
 ضلعه عن مربع قطر الاقصر بقيت مساحة **طرق** **احد** تضعف
 مربع احد اضلاعه ويزيد عليه حاصل ضرب جذره في ضعف احد اضلاعه
 فهو المط واما استخراج ابعاده فنضعف مربع احد اضلاعه و
 نزيد جذره على احد اضلاعه يحصل قطر الاقصر واذا كان قطر الاقصر
 معلوما والضلع مجهولا تضعف مربع قطر الاقصر فمأخذ جذر الحاصل
 وينقص منه قطر الاقصر فبقي فهو ضلع منه **الباب**
الرابع في مساحة الدائري وابعاضها عن القطاع والتقوس
 الكلتة وغير ذلك وما يتعلق بها وهو مشتمل على خمسة فصول
الفصل الاول في النونيات الدائرة سطح مستوي كسطح
 مستدير وفي داخله نقطة يكون جميع الخطوط المستقيمة الخارجة



by twenty seven, then half the [square] root of the result to get the area.

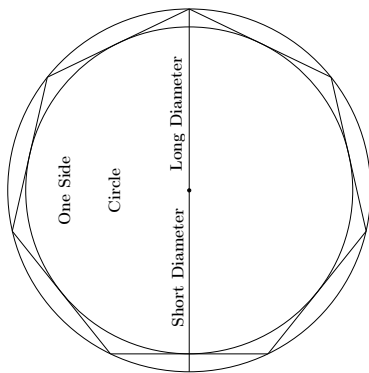
Another Method. We multiply the fourth power of the radius of the inner-circle by twelve, then take the [square] root of the result and that is the desired [area].



Another Method. We multiply the cube of one side by the sum of all sides and add one eighth of the result to itself. This results in the square of the area. Since the hexagon is equivalent to six equilateral triangles, its side is the same as the side of the triangle.

As for obtaining its dimensions [sides], we take the square root of three times its side squared. This gives the shorter diameter which is twice the height of the equilateral triangle that is one sixth of the hexagon. The larger diameter is twice its side.

Fifth Section. Matters specific to regular octagon not discussed before and the extraction of its dimensions. As for the area, we subtract the square of its side from the square of the shorter diameter. What is left is its area.



Another Method. We double the square of one of its sides, and add it to the result of the product of its square root and twice one of its sides. The output is the desired [area].

As for obtaining its dimensions [sides], we double the square of one of its sides, and add its square root to one of its sides. This gives the shorter diameter. If the shorter diameter is known and the side is unknown, we double the square of the shorter diameter and take the root of the result from which we subtract the shorter diameter. What remains is the side.

Fourth Chapter On the area of a circle and its parts

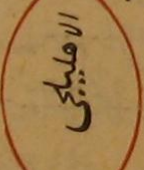
I mean portions, sectors, annulus and other regions, and related matters. It contains five sections.

First Section. On Definitions. The circle is a flat surface enclosed by a round curve. Inside of it there is a point such that all straight line segments from that point to

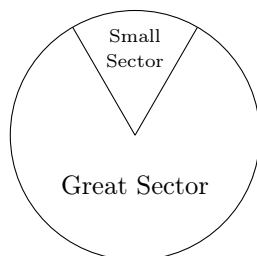
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عنها اليه متساوية وذلك الخط محيطها وتلك النقطة مركزها والخطوط
 الخارجة انصاف اقطارها وكل خط مستقيم يقطع الدائرة
 بتسعين فيقال لما وقع منه فيها وتر وما يبرز من المحيط قوس
قطاع الدائرة سطح يحيط به قوس من محيط الدائرة وقطبان متساويين
 بهما نصفاً قطرتك الدائرة يلتقيان عند مركزها قطعه الدائرة
 سطح يحيط به قوس اقل من النصف او اكثر
 وخط مستقيم واصل بين طرفي القوس
 اعني وتر تلك القوس يقال له قاعدة
 القطعة ونصف وتر القوس جيب لنصف ذلك القوس
 والعمود الخارج من منتصف القوس على منتصف وترها
 لذلك القوس عند بعض ونصف القوس عند الاكثرين
الاهليلجي هو المحاط بقوسين متساويين من دائرتين
 متساويتين كل منهما اصغر من نصف المحيط وان كانا اكثر فقسمة
 بالشجيم وصورتها هكذا
اطلقة المسطحة
 سطح يحيط به محيط
 دائرتين مركزهما واحد واذا قطعت بخطين متساويين بالمرکز فتنسج
 كل واحد من قطعها بقطعة الخلقه الهلال سطح مستوي
 به قوسان ليا اكثر من النصف من دائرتين اما متساويتين



the circle are equal. That curve is the circumference of the circle. The point is its center, and the line segments from that point are the radii of the circle. Each straight line divides the circle into two parts. The part that falls inside the circle is called a chord, and the part that is from the circumference is called an arc. A sector of the circle is the region enclosed by an arc of the circle from the circumference and two equal lines that are radii of that circle intersecting at the center.

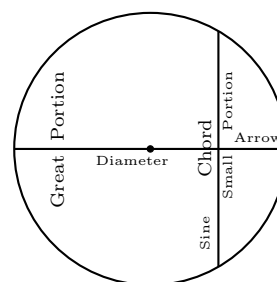
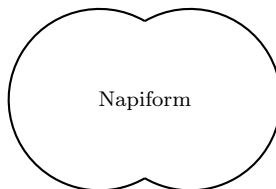


A portion of the circle is the region enclosed by an arc less than or greater than the semicircle and a straight line connecting the two ends of the arc, i.e., the chord of that arc which is called the base of the portion. Half of the chord of an arc is the sine of half of that arc. ⁴⁴

The outer height from the midpoint of the arc to the midpoint of the chord is, according to some people, the arrow of that arc, and according to most people it is [the arrow] of half of the arc. ⁴⁵ An elliptical region ⁴⁶ is one enclosed by two equal circular arcs each of which is smaller than half of the circumference. If the circular arcs are greater than half of the circumference then the shape is called napiform. ⁴⁷

Their figures are as follows.

A flat annulus is the region enclosed by two eccentric circles. If it is cut by two lines passing through the center then each resulting part is called a section of the annulus.



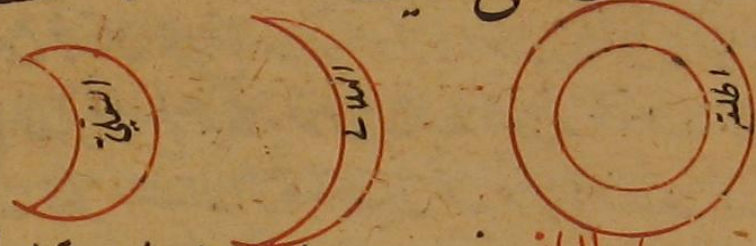
A crescent is the flat region enclosed by two arcs neither of which is greater than half their circles, whether the circles are equal

⁴⁵ If θ is the central angle of a unit circle, the arrow which is sometimes called sagitta is $1 - \cos(\theta/2) = \text{versin}(\theta/2)$.

⁴⁶ Lens

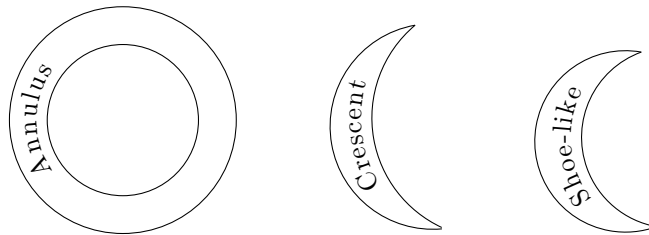
⁴⁷ A shape like a turnip. The word “shalgam” used in the manuscript most likely in Persian or Turkic languages means turnip.

او مختلفتين محدبهما الى جهة واحدة وما كان كل واحد من القوس
اكثر من النصف سمي تعلية وصورتها هكذا



الفصل الثاني في مساحة الدائرة واستخراج المحيط
عن القطر وبالعكس ولتقدم في هذا الفصل ثم نشرع
في المسألة اعلم ان المحيط ثلثة امثال القطر وكسره وهو اقل من سبع
القطر لكن القوم اخذوه سبعة بسهولة الحساب وقال اريستيدس
ان ذلك الكسر اقل من السبع واكثر من عشرة اجزاء من احد
سبعين وعلى ما حصلناه وذكرناه في رسالتنا المسماة بالمحيط
وهو $7 \frac{1}{2}$ كط مد ثلثة بعد طرح الروابع وما بعد ما اذا كان القطر
واحدا وهذا ادق من حساب اريستيدس بكثير على ما بيناه
في الرسالة المذكورة واقرب منه الى الصواب لكنه باثباته
لا يعرفه الا اية تبارك وبها فاذا كان قطر دائرة معلوما ومحيطها
مجهولا فنضرب القطر في ذلك العدد ليحصل المحيط وان كان
بالعكس قسم المحيط على ذلك العدد لينجى القطر وان كانا
مجهولين نضع على المحيط نقطتين كيف اتفق وندير عليهما
دائرتين متساويتين بحيث يتقاطعا ونصل بين هذين التقاطعين

or not. It is concave in one direction. If each of the two arcs is greater than half [of the circle] then it is called shoe-like.⁴⁸ And these are their images:



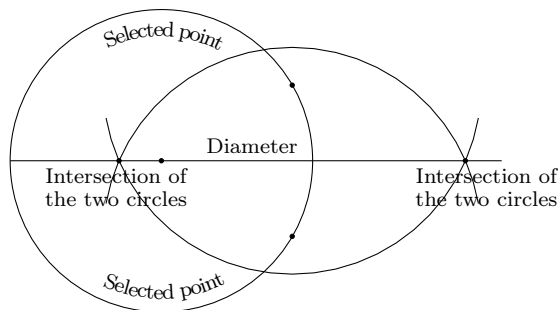
Second Section. On the area of a circle, finding the circumference from the diameter, and vice versa. We start with [calculating the circumference] in this chapter, then the area. Know that the perimeter is three times the diameter and a fraction which is less than one seventh of the diameter but folks take it as seventh for the simplicity of calculation. Archimedes said that this fraction is smaller than a seventh and greater than ten out of seventy one. According to what we obtained and mentioned in our article titled “Muhitiyya/Circumference” it is $3;8 : 29 : 44$ thirds⁴⁹ after dropping fourths and what comes after, if the diameter is one. This is far more accurate than the calculation of Archimedes as we explained in the article mentioned, and it is closer to the correct value. However, nobody knows it exactly⁵⁰ except for God, The Almighty. If the diameter of the circle is known and the perimeter is unknown then we multiply the diameter by that number to obtain the perimeter. In the opposite situation, we divide the perimeter by that number [π] to obtain the diameter. If both are unknown then we pick two points on the circumference, however we wish, and we draw two equal circles centered at those points so that they intersect. We connect the two points of intersection

⁴⁸ Horseshoe

⁴⁹ Here, al-Kāshī is giving an approximation to the number π in the sexagesimal system. His actual approximation in the article, which was by far the best of his time, was more accurate than this.

⁵⁰ Since π is an irrational number, it is not possible to write its exact numerical value using a finite number of digits.

by a straight line. If we extend this line in both directions so that it intersects with the circumference, then the diameter is obtained as follows:



If the area is known then we multiply it by 14 and divide the outcome by 11. We then take the square root of this result which is the diameter. Or, we multiply the area by seven, then divide the result by 22 and take its square root to obtain the radius. These two are well known calculations. As for our calculation, we divide the area by $3;8 : 29 : 44$ thirds⁵¹, then take the root of the result. That is the radius. If we divide the area by $0;47 : 7 : 26$ third⁵² then take the root of the result, we get the diameter.

We have a trick to find the length of the circumference. Put a string on the circumference, then measure the circumference. Or put one of the two ends of the string at a point on the circumference, then we move it so that it is tangent to a small portion of the circumference until we find the total. To obtain the area, we multiply the radius by half of the circumference.

Another Method. We multiply the square of the radius by the ratio of the circumference to the diameter, that is, by three and a seventh in the well known calculation, or we multiply it by 22 and divide the result by seven, in our calculation by $3;8 : 29 : 44$ thirds. What results is the area.

Another Way. We multiply the square of the diameter by eleven, then divide the result by fourteen. This gives the area by a well known calculation. In our calculation we multiply by $0;47 : 7 : 26$ thirds, which is the ratio of the area to the square of the diameter

[and it yields what we want. This number is a quarter of the first number because the ratio of the area of the circle to the square of the radius is the same as the ratio of the first number which is $3;8 : 29 : 44$ thirds to one. And the ratio of the square of the radius to the square of the diameter is a quarter. We put the results of the products of these two numbers by sexagesimal numbers in a table to facilitate the operations. We also converted it to the decimal number system. We made its fractions into decimal fractions. The tables are as follows:]⁵³

⁵¹ This is π in sexagesimal which corresponds to 3.14159259259 in the decimal system.

⁵² This is $\pi/4$ in sexagesimal numbers.

⁵³ This is found on the left margin with an indication to insert at the end of the text.

Table of the change of ratios of the circumference to the diameter												Table of change of ratios of the area of a circle to the square of the diameter											
Diameter	Perimeter						Diameter	Perimeter					Square of the Diameter	Area				Square of the Diameter	Area				
	Elevate	Parts	Minutes	Seconds	Thirds			Elevate	Parts	Minutes	Seconds	Thirds		Parts	Minutes	Seconds	Thirds		Parts	Minutes	Seconds	Thirds	
1	0	3	8	29	44		31	1	37	23	21	44	1	0	47	7	26	31	24	20	50	26	
2	0	6	16	59	28		32	1	40	31	51	28	2	1	34	14	52	32	25	7	57	52	
3	0	9	25	29	12		33	1	43	40	21	12	3	2	21	22	18	33	25	55	5	18	
4	0	12	33	58	56		34	1	46	48	50	56	4	3	8	29	44	34	26	42	12	44	
5	0	15	42	28	40		35	1	49	57	20	40	5	3	55	37	10	35	27	29	20	10	
6	0	18	50	58	24		36	1	53	5	50	24	6	4	42	44	36	36	28	16	27	36	
7	0	21	59	28	8		37	1	56	14	20	8	7	5	29	52	2	37	29	3	35	2	
8	0	25	7	57	52		38	1	59	22	49	52	8	6	16	59	28	38	29	50	42	28	
9	0	28	16	27	36		39	2	2	31	19	36	9	7	4	6	54	39	30	37	49	54	
10	0	31	24	57	20		40	2	5	39	49	20	10	7	51	14	20	40	31	24	57	20	
11	0	34	33	27	4		41	2	8	48	19	4	11	8	38	21	46	41	32	12	4	46	
12	0	37	41	56	48		42	2	11	56	48	48	12	9	25	29	12	42	32	59	12	12	
13	0	40	50	26	32		43	2	15	5	18	32	13	10	12	36	38	43	33	46	19	38	
14	0	43	58	56	16		44	2	18	13	48	16	14	10	59	44	4	44	34	33	27	4	
15	0	47	7	26	0		45	2	21	22	28	0	15	11	46	51	30	45	35	20	37	0	
16	0	50	15	55	44		46	2	24	30	47	44	16	12	33	58	56	46	36	7	41	55	
17	0	53	24	25	28		47	2	27	39	17	28	17	13	21	6	22	47	36	54	49	22	
18	0	56	32	55	12		48	2	30	47	47	12	18	56	32	55	12	48	37	41	56	48	
19	0	59	41	24	56		49	2	33	56	16	56	19	14	55	21	14	49	38	29	4	14	
20	1	2	49	54	40		50	2	37	4	46	40	20	15	42	28	40	50	39	16	11	40	
21	1	5	58	24	24		51	2	40	13	16	24	21	16	29	36	6	51	40	3	19	6	
22	1	9	6	54	8		52	2	43	21	46	8	22	17	16	43	32	52	40	50	26	32	
23	1	12	15	23	52		53	2	46	30	15	52	23	18	3	50	58	53	41	37	33	58	
24	1	15	23	53	36		54	2	49	38	45	36	24	18	50	58	24	54	42	24	41	24	
25	1	18	32	23	20		55	2	52	47	15	20	25	19	38	5	50	55	43	11	48	50	
26	1	21	40	53	4		56	2	55	55	45	4	26	20	25	13	16	56	43	58	56	16	
27	1	24	49	22	48		57	2	59	4	14	48	27	21	12	20	42	57	44	46	3	42	
28	1	27	57	52	32		58	3	2	12	44	32	28	21	59	28	8	58	45	33	11	8	
29	1	31	6	22	16		59	3	5	21	14	16	29	22	46	35	34	59	46	20	18	34	
30	1	34	14	52	0		60	3	8	29	44	0	30	23	33	43	0	60	47	7	26	0	

[illegible]

		Fractions						
		Integers	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths	Millionths
Change of ratios of circumference to diameter	1	03	1	4	1	5	9	3
	2	06	2	8	3	1	8	6
	3	09	4	2	4	7	7	9
	4	12	5	6	6	3	7	2
	5	15	7	0	7	9	6	5
	6	18	8	4	9	5	5	7
	7	21	9	9	1	1	5	1
	8	25	1	3	2	7	4	2
	9	28	2	7	4	3	3	7
	10	31	4	1	5	9	3	0

		Fractions								
		Integers	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths	Millionths	Ten-millionths	Hundred-millionths
Change of ratios of the area of circle to the square of diameter	1	0	7	8	5	3	9	8	2	5
	2	1	5	7	0	7	9	6	8	0
	3	2	3	5	6	1	9	4	7	5
	4	3	1	4	1	5	9	3	0	0
	5	3	9	2	6	9	9	1	2	5
	6	4	7	1	2	3	8	9	5	0
	7	5	4	9	7	7	8	7	7	5
	8	6	2	8	3	1	8	6	0	0
	9	7	0	6	8	5	8	4	2	5
	10	7	8	5	3	9	8	2	5	0

مثال مساحه دائره يكون نصف قطرها سبعة وسبعين ذراعاً فيما
 فهمت عليه القوم ضربناه في **أ** بان ضربناه في الكسر المجنس وهو
م حصل **م** قسمناه على المخرج وهو سبعة فخرج من القسمة **م**
 وهو نصف المحيط تقريباً او بان نضرب تارة في الثلثة حصل **م**
 وتارة في السبع حصل **أ** جمعناهما بلغ وهو نصف المحيط وان كان المحيط
 معلوماً وارادنا معرفة نصف القطر ضرب نصف المحيط وليكن **م**
م في **م** بان نضرب في الكسر وهو سبعة وقسمنا اياه على
م المخرج فخرج من القسمة **م** وهو نصف القطر ضربنا نصف
 القطر في نصف المحيط حصل **م** وهو المساحه **طريقه اخرى**
 نربع القطر وهو **م** حصل **م** نضرب في **أ** حصل
م قسمناه على **م** فخرج من القسمة **م** مطابقاً للاول
 ثم عملنا بالرقوم الجمل هكذا ضربنا نصف القطر وهو **أ** ذراعاً
 في **ب** حصل **م** قسمناه على **أ** اذا كانت نسبة القطر الى المحيط
 حسب مدعاتهم نسبة السبعة الى اثنين وعشرين فخرج من القسمة
م ذراعاً وهو نصف المحيط ضربناه في نصف القطر حصل
م ذراعاً وهو حرقوع ذراعان المساحه مطابقاً للاول اما على
 ما استقصينا فيه ضربنا **أ** نصف القطر في نسبة المحيط الى القطر
 بان فعلنا في **أ**

Example. The area of a circle with a radius of seventy-seven cubits. Following the common method, we multiply it by $\frac{3}{7}$. That is, by multiplying by the numerator of the combined fraction 22. The result is 1694. Then, we divide the result by the denominator which is seven. The result of the division is 242 which is approximately half of the circumference. Or, we first multiply it [the radius] by three and get 231, then multiply it [the radius again] by one seventh and get 11. We add the two numbers and obtain [242]⁵⁴ which is the half of the circumference. If the circumference is known and we want to know the radius, then we multiply half of the circumference, let it be 242, by $\frac{0}{22}$. We first multiply it by the numerator which is seven, then divide the result by the denominator which is 22. The result of the division is 77 which is the radius. Then, we multiply the radius by half the circumference. It yields 18634 which is the area.

Another Method. We square the diameter which is 154, and the result is 23716. We multiply it by 11 and get 260876. Then we divide it by 14 to get 18634 which agrees with the earlier calculation. Next, we work in the jummal system⁵⁵ as follows. We multiply the radius which is 1 : 17 cubit by 22. The result is 28 : 44 which we divide by 7 if the ratio of the diameter of a circle to its circumference is equal to the ratio of seven to twenty-two as they claim. The result of the division is 4 : 2 cubits which is half of the circumference. We multiply it by the radius and obtain 5;10 : 34 cubits. This converts to the same measure of the area in decimal in agreement with the earlier calculation. As for what we have investigated, we multiply 1 : 17 which is the radius, by the ratio of the circumference to the diameter. These calculations are given in the following table:

⁵⁴ Found on the right margin with an insert symbol.

⁵⁵ Literally meaning “numerals of sentences”, *jummal system/notation* is a way to represent sexagesimal numbers in which the letters of the Arabic alphabet are used as digits. It was commonly used in astronomical tables.

In the table we take what is next to 1 to get	0	3	8	29	44	
Then we take what is next to 17			53	24	25	28
We add them up to get half the circumference		4	1	54	9	28
We multiply it by 1 : 17 to get the area	5	10	26	30	8	56
	Second	Elevate				
		Elevate	Cubit	Minute	Second	Third
	Cubits		Fractions			

This area is more accurate than what we obtained by the famous calculation method and it is approximately seven and a half cubits less than it.

Another Method. Square the diameter and it becomes 6;35 : 16 which we multiply by the ratio of the [area of the] circle to the square of the diameter and obtain 5 : 10 : 26;30 : 8 : 56 thirds. If the area is known and we want to know the diameter, then we divide it, which is what is mentioned before, by 47 : 7 : 26 thirds. We carry out the operation using the following table:

	5	10	26	30	8	56
6	4	42	44	36		
		27	41	54		
35		27	29	10	10	
			12	33	58	
16			12	33	58	56

The result of the division is 6;35 : 16. We take the square root of that and it becomes 2;34 which is one hundred fifty four. The operation in Indian numeral is as follows:

In the table we take what is next to 7 and get				2	1	9	9	1	1	5	1
Then we take what is next to 7 by ten			2	1	9	9	1	1	5	1	
We add them up to get half of the circumference			2	4	1	9	0	2	6	6	1
We multiply it by 77 to get the area	1	8	6	2	6	5	0	4	8	9	7
	Ten-thousands	Thousands	Hundreds	Tens	Unit	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths	Millionths
	Integers					Fractions					

Another Method. Suppose the square of the diameter is 23716. We take what is next to each of its digits from the table of the ratio of the area to the square of the diameter, and we put them step by step as follows.

2	1	5	7	0	7	9	6	9	0				
3		2	3	5	6	1	9	4	7	5			
7			5	4	9	7	7	8	7	7	5		
1					7	8	5	3	9	8	2	5	
6					4	7	1	2	3	8	9	5	0
	1	8	6	2	6	5	0	4	8	9	7	0	0
Integers					Fractions								

We laid the explanation on how to operate on this table in our article called "The Circumference."

Third Section. On the area of sectors, portions, and finding dimensions from each other.

For the area, we multiply the length of the radius by the measure of half of the arc.

Another Method. We obtain the area of the sector of a circle by multiplying the measure of the arc of the sector measured in degrees where the circumference is three hundred sixty, called the circumferential degrees, by the sixth of the area of that circle.⁵⁶

Another Method. We multiply the length of the radius by the measure of half of its arc in units so that the radius is sixty and the circumference is approximately three hundred seventy seven. [Now if you multiply the radius which is sixty by the three and a seventh you get the circumference which is approximately three hundred seventy seven]⁵⁷. If we project on a circle the triangle of a sector which is smaller than half of the circle we get the smaller portion. If we add it to the one that is larger than half, then the larger portion is obtained.

As for obtaining some of the dimensions from others, if the radius and the chord are known in terms of the same unit and if we want to know the measure of its arc, we divide half of the chord by its radius reduced and we take the arc-sine of the result. What is obtained is half of its arc in terms of degrees that divide the circumference into three hundred sixty. If we add to it the third of its seventh

⁵⁶ If θ in degrees is the central angle of a sector of radius r , then, the area of the sector is $\pi r^2 \left(\frac{\theta}{360} \right) = \left(\frac{\theta}{60} \right) \left(\frac{1}{6} \pi r^2 \right)$.

⁵⁷ Found in the right margin with an indication to insert in the text.

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لحساب المشهور ونضرب ثلثه في نسبة المحيط الى القطر الذي
 وضعنا لما في الجدول فمما حصل وهو مقدار نصف قوسه بالاجزاء
 التي بها نصف القطر ستون ثم اذا ضربناه في درعان نصف
 القطر حصل درعان نصف المحيط ولو نضرب درعان نصف
 القطر في نسبة المحيط الى القطر وهو بحسبنا يخرج القطر مد و١٠
 المشهور ثلثه وسبع ونضرب الكااصل في مقدار نصف قوسه بمائة
 المحيط ثلثمائة وستون ونقسم الكااصل على مائة وثمانين يخرج درعان
 نصف القوس وان كان نصف القطر والسهم معلومين
 والباقي بمقدار نقص السهم عن نصف القطر فابقي وهو
 العمود الخارج عن زاوية القطاع على منتصف الوتر نريد
 على نصف القطر ونضرب المجموع في السهم وناخذ جذرا الكا
 وهو نصف وتره والباقي كما سبق قال جامع للمجموع قطا
 كان نصف قطره اثني عشر وسهم اثنين نقضنا الاثنين عن
١٢ بقى ٢٥ زدناه على ١٢ بلغ ٣٧ ضربناه في ٢ حصل ٧٤
 اخذنا جذره فكان ٨٦ قسمناه على نصف القطر من خطا خرج ٨٦
 وهو جيب نصف قوسه قوسناه فصار ٨٦ وهو نصف القوس
 بالاجزاء التي بها المحيط ثلثمائة وستون اخذنا ثلث سبعة لحساب المشهور
 بان قسمناه على اتما فكان ١٥٦ زدناه عليه بلغ ٤٦٨ ثمانية
 وهو نصف قوسه بالاجزاء التي بها نصف القطر ستون وكما

according to well-known calculation or multiply its third by the ratio of the circumference to the diameter which we recorded in the table. What is obtained is the measure of half of its arc in the units where the radius is sixty. If we then multiply it by the length of the radius we obtain the length of half the circumference. If we multiply the length of the radius by the ratio of the circumference to the diameter, which according to our calculation is $3;8 : 29 : 44$ and according to the well known calculation three and one seventh, and multiply the output by the measure of half its arc in which the circumference is divided into three hundred sixty, and divide the output by hundred eighty, the measure of half of the arc is obtained. If the radius and the arrow are known and the rest are unknown, we subtract the arrow from the radius. What remains is the height from the angle of the sector on the middle of the chord. We add it to the radius, multiply the sum by the arrow, and take the square root of the result. That is half its chord. The rest is as before.

An encapsulating Example. A sector whose radius is twelve and whose arrow is two. We subtract the two from 12, what remains is 10. We add it to the 12 and get 22. We multiply it by 2 and get 44. We take its square root and get $6;38$. We divide it by the radius reduced [by a factor of 60], the result is $33;10$, which is the sine of half its arc. We take its arc-sine and it becomes $33 : 22$, which is half the arc measured in units in which the circumference is three hundred and sixty. We take a third of its seventh according to the well-known calculation. We do that by dividing it by 21 to get $1;35 : 20$. We add this value to the half arc and get $34;58 : 20$, which is half of its arc measured in units in which the radius is sixty. According to our calculations,

we multiply the third of 33;22 which is 11;08 : 20 by 3;08 : 29 : 44 and get 34;56 : 29 : 22 third. This is half the arc measured in units in which the radius is sixty. We multiply it by the radius, which is 12. Using the well-known calculation, we get 6;59 : 28, which is the length of half of its arc. And according to our calculations, we get 6;59 : 17 : 52 thirds.

Another method. We multiply the radius which is 12 by three and a seventh according to the well-known calculation. We get $\frac{37}{5}$, which is 37;42;51 in the jummāl system. We multiply it by half the arc measured in circumferential units which is 33;22, we get 20;58 : 24 seconds. We divide it by one hundred eighty and get 6;59 : 28, which is the length of half the arc in the well-known calculation agreeing with the previous [answer]. According to our calculations, we multiply 12 by 3;08 : 29 : 44 to get 37;41 : 56 : 48. We multiply it by 33;22 and get 20;57 : 53. We divide it by one hundred eighty. The result of the division is 6;59 : 17 : 52 thirds, just like what preceded. If the chord and arrow are known and the rest are unknown, we divide the square of half the chord by the arrow. We add the result to the arrow and take half of the sum to get the radius. If the length of the chord is known, and so is the arc, in circumferential units, we divide half the chord by the sine of half the arc reduced [by a factor of 60]. The result is the length of the radius. And, if only the lengths of the arc and the chord are known and we want to know the radius, then we can do that either by hand or by reading the sine table when the ratio of that sine to its arc is equal to the ratio of the known chord to the known arc. That arc is half of the sector's arc in degrees where the circumference is three hundred sixty.

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وان كان ذراعان القوس ونصف القطر معلومين وارادنا معرفة الوتر
لما د القطعة ضرب نصف القوس في نسبة المحيط الى القطر ونقسم
عليه حاصل ضرب نصف القوس في مائة وثمانين فما خرج فهو نصف
القوس بمائة المحيط ثلثمائة وستون ضرب جيبه في ذراعان نصف القطر
منحليما فما حصل فهو ذراعان نصف الوتر واعلم ان القطاع الذي
يكون قوسه ربع دائرة او يعلوها اذا وقعت في دائرة بحيث تماس
طراف قوسه محيط الدائرة فالقطاع نصف تلك الدائرة والدائرة
التي وقعت في القطاع ارشئ يكون نسبتها الى ذلك القطاع نسبة
الواحد الى **ع ل ط م ح م** ونصف قطر **ع م د** بالاجزاء التي بها نصف
قطر القطاع ستون **الفصل الرابع** في مساحات السائر السطوح
التي يحيط بها الخطوط المستديرة ما ذكرنا وما مساحته الا هليلجي في
مجموع مساحات القطعتين الحاصلتين عن جيب قطره الاطول ومساح
الهلال والنفلي هي الفضل بين القطعتين اذا توهم
خط وصل بين طرفيهما واما السطح الذي يحيط به قوسان من
دارتين مختلفتين متحدتهما اما في جهتين مختلفتين كالسطح المنكسف
او المنكسف من صفتي النيرين في المنكسفات والمنكسفات
البحرية واما في جهة واحدة كالنوراني الباتية منها واذا كان نصف
قطرهما وقطره الاصغر معلوما فقط فطريق مساحته ذكرناه في زيجنا
المسمى بالنزج الكافان فمن اراد معرفة فعلية الرجوع الى ذلك

And if the length of the arc and the radius are known and we want to find the chord for [calculating] the area of a sector, we multiply the radius by the ratio of the circumference to the diameter, then we divide by it the product of half the arc and one hundred eighty. The result is half the arc in degrees where the circumference is three hundred sixty. We multiply its sine by the length of the radius reduced, the result is the length of half the chord. Know that for the sector whose arc is a quarter of a circle or its third, if it falls in a circle whose circumference passes through both sides of its chord and its vertex, then the sector is half of that circle. And the circle that falls in the quarter sector has a ratio to that sector equal to the ratio of 1 to 0;39 : 58 : 46, and its radius is 0;24 : 51 : 10 in the unit in which the radius of the sector is sixty.

Fourth Section. On the area of all surfaces enclosed by curved lines that we mentioned. As for the area of an elliptical [shape], it is the sum of the areas of the two portions on either side of its longer diameter. The area of a crescent and shoe-like is the difference between the portions if you were to assume a line connecting their ends. Consider the surface enclosed by two arcs from two different circles, whether they are convex with different directions like the dark surface in partial lunar and solar eclipses, or in the same direction, like the bright side of them. If their [the two circles'] radii and its [the surface's] smaller radius were the only known [parts], then we have mentioned the way to calculate its area in our Zij called the Khaqani Zij. So, whoever wants to know the method should refer to it.

وامتاساحة الكلبة المنسطح مني فصل مساحه الدائرة العظمى
 على الدائرة الصغرى او حاصل ضرب البعد بين الدائرتين في نصف
 مجموع محيط الدائرتين ومساحة قطعه الكلبة المنسطح مني حاصل ضرب
 نصف مجموع القوسين المحيطين بها في البعد بين القوسين ■
الفصل الخامس في جدول الجيب وكيفيه العمل به انما خذ
 بازار درجات القوس من الجدول جيبها وان كانت معها دقائق
 نضربها في تفاضل السطرين ونضع الحاصل تحت جيب الدرجات
 منخطا بمرتبة وان كانت معها ثوان نضربها في التفاضل المذكور
 ايضا ونضع الحاصل تحت حاصل الدقائق منخطا بمرتبة اخوة
 ثم نجمع الجميع يحصل جيب تلك القوس وقد وضعنا تفاضلا
 بين السطرين لكل جيب بازاوية في جدول **افترس** اردنا

جيب ٨٥	٧ ٥	نكان	افترس بازاو قوس ٨٥
١٠٠	١٠٠	١٠٠	وكان اتفاضل بازاو ٨٥
١٠٠	١٠٠	١٠٠	له ضربناه في ٨٥
١٠٠	١٠٠	١٠٠	حصل
١٠٠	١٠٠	١٠٠	وضربناه في ذلك اتفاضل ايضا حصل
١٠٠	١٠٠	١٠٠	جمعنا فصار الجيب المطلوب

وان كان معنا جيب وزيد قوسه نطلب في الجدول اكثر جيب
 يمكن نقصانه عن الجيب المحفوظ فاذا وجد نقصه منه ويكتب قوسه
 اعني العدد الموضوع بازاوية على حاشية الجدول وجه الدرجات
 وما بقى من الجيب نقصه على تفاضلا بين السطرين فما خرج فهو

The area of an annulus is the difference between the area of the bigger circle and the area of the smaller circle, or the product of the distance between the two circles and half the sum of the circumferences of the two circles. The area of the portion of an annulus is the result of the multiplication of half the sum of the two arcs enclosing it by the distance between the two arcs.

Fifth Section. The sine table and how to use it. We take the sine of the degrees of the arc, and if it has minutes, then we multiply them by the difference between the two lines, then we put the result underneath the sine of the degrees reduced by one position. And if it has seconds, then we multiply it by the aforementioned difference as well, and we put the result underneath the result of the minutes reduced by another position. Then, we add them all to get the sine of that arc. We have also put the difference between each two lines for each sine next to it in another table.

Example. We want the sine of 15;21 : 48.

We take what is aligned with the arc 15, we get	15;31 : 45
The difference aligned with it is 60;32. We multiply it by 21 to get	0;21 : 12
We multiply 48 by that difference as well to get	0;0 : 48
We add them up to get the required sine	15;53 : 45

And if we have the sine and we want its arc, then we look up in the table the biggest sine that can be subtracted from the given sine. When we find it, we subtract it from the given sine and get its arc, i.e, the number written next to it on the margin of the table, which is the degrees. Then we divide the remainder of the sine by the difference between the two lines, and what results is

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دقایق القوس و ثوابها **مثله** كان معنایب وهو **له**
 واردا قوسه و طلبنا في الكدول اكثر جیب يمكن نقصانه عنه
 فوجدنا بازائه من الدرجات **له** لا **له** من الجیب نقصناه عن
 الجیب المخفض اعني **له** **له** بقي **له** **له** قسمناه على
 تفاضل بين السطرين وهو كان **له** **له** خرج من القسمة من الدقائق
 والثواني **له** **له** جمعناه مع الدرجات فصار **له** **له**
 وهو القوس المط **له** **له** اراد التدقيق فعليه الرجوع بجدول
 الزيج الالمانی في اوزینها المعروف بالحقات اذ كان هذا المقدار
 كافيا في هذا الكتاب **له**

والكدول

له

the minutes of the arc and its seconds.

Example. Let us have a sine which is $15;53 : 45$ and we want its arc. We find the largest sine that can be subtracted from it, and we find it next to 15 degrees, with a value of $15;31 : 45$. We subtract it from the given sine, i.e $15;53 : 45$, what remains is $0;22$. We divide that by the difference between the two lines which is $60;32$. The result of the division is $21 : 48$ in minutes and seconds. We add them to the degrees and get $15;21 : 48$, which is the wanted arc. Whoever wants to check can refer to the table of the Ilkhani Zij, or our Zij known as the Khaqani, as this amount [of explanation] is enough for this book. Here is the table [of sine]:

جدول الجيب									
الارتفاع	العرض	الارتفاع	العرض	الارتفاع	العرض	الارتفاع	العرض	الارتفاع	العرض
١	١	١	١	١	١	١	١	١	١
٢	٢	٢	٢	٢	٢	٢	٢	٢	٢
٣	٣	٣	٣	٣	٣	٣	٣	٣	٣
٤	٤	٤	٤	٤	٤	٤	٤	٤	٤
٥	٥	٥	٥	٥	٥	٥	٥	٥	٥
٦	٦	٦	٦	٦	٦	٦	٦	٦	٦
٧	٧	٧	٧	٧	٧	٧	٧	٧	٧
٨	٨	٨	٨	٨	٨	٨	٨	٨	٨
٩	٩	٩	٩	٩	٩	٩	٩	٩	٩
١٠	١٠	١٠	١٠	١٠	١٠	١٠	١٠	١٠	١٠
١١	١١	١١	١١	١١	١١	١١	١١	١١	١١
١٢	١٢	١٢	١٢	١٢	١٢	١٢	١٢	١٢	١٢
١٣	١٣	١٣	١٣	١٣	١٣	١٣	١٣	١٣	١٣
١٤	١٤	١٤	١٤	١٤	١٤	١٤	١٤	١٤	١٤
١٥	١٥	١٥	١٥	١٥	١٥	١٥	١٥	١٥	١٥
١٦	١٦	١٦	١٦	١٦	١٦	١٦	١٦	١٦	١٦
١٧	١٧	١٧	١٧	١٧	١٧	١٧	١٧	١٧	١٧
١٨	١٨	١٨	١٨	١٨	١٨	١٨	١٨	١٨	١٨
١٩	١٩	١٩	١٩	١٩	١٩	١٩	١٩	١٩	١٩
٢٠	٢٠	٢٠	٢٠	٢٠	٢٠	٢٠	٢٠	٢٠	٢٠
٢١	٢١	٢١	٢١	٢١	٢١	٢١	٢١	٢١	٢١
٢٢	٢٢	٢٢	٢٢	٢٢	٢٢	٢٢	٢٢	٢٢	٢٢
٢٣	٢٣	٢٣	٢٣	٢٣	٢٣	٢٣	٢٣	٢٣	٢٣
٢٤	٢٤	٢٤	٢٤	٢٤	٢٤	٢٤	٢٤	٢٤	٢٤
٢٥	٢٥	٢٥	٢٥	٢٥	٢٥	٢٥	٢٥	٢٥	٢٥
٢٦	٢٦	٢٦	٢٦	٢٦	٢٦	٢٦	٢٦	٢٦	٢٦
٢٧	٢٧	٢٧	٢٧	٢٧	٢٧	٢٧	٢٧	٢٧	٢٧
٢٨	٢٨	٢٨	٢٨	٢٨	٢٨	٢٨	٢٨	٢٨	٢٨
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٣٢	٣٢	٣٢	٣٢	٣٢	٣٢	٣٢	٣٢	٣٢	٣٢
٣٣	٣٣	٣٣	٣٣	٣٣	٣٣	٣٣	٣٣	٣٣	٣٣
٣٤	٣٤	٣٤	٣٤	٣٤	٣٤	٣٤	٣٤	٣٤	٣٤
٣٥	٣٥	٣٥	٣٥	٣٥	٣٥	٣٥	٣٥	٣٥	٣٥
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٣٧	٣٧	٣٧	٣٧	٣٧	٣٧	٣٧	٣٧	٣٧	٣٧
٣٨	٣٨	٣٨	٣٨	٣٨	٣٨	٣٨	٣٨	٣٨	٣٨
٣٩	٣٩	٣٩	٣٩	٣٩	٣٩	٣٩	٣٩	٣٩	٣٩
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٤٢	٤٢	٤٢	٤٢	٤٢	٤٢	٤٢	٤٢	٤٢	٤٢
٤٣	٤٣	٤٣	٤٣	٤٣	٤٣	٤٣	٤٣	٤٣	٤٣
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٤٧	٤٧	٤٧	٤٧	٤٧	٤٧	٤٧	٤٧	٤٧	٤٧
٤٨	٤٨	٤٨	٤٨	٤٨	٤٨	٤٨	٤٨	٤٨	٤٨
٤٩	٤٩	٤٩	٤٩	٤٩	٤٩	٤٩	٤٩	٤٩	٤٩
٥٠	٥٠	٥٠	٥٠	٥٠	٥٠	٥٠	٥٠	٥٠	٥٠

مس ٥ ٥

البار

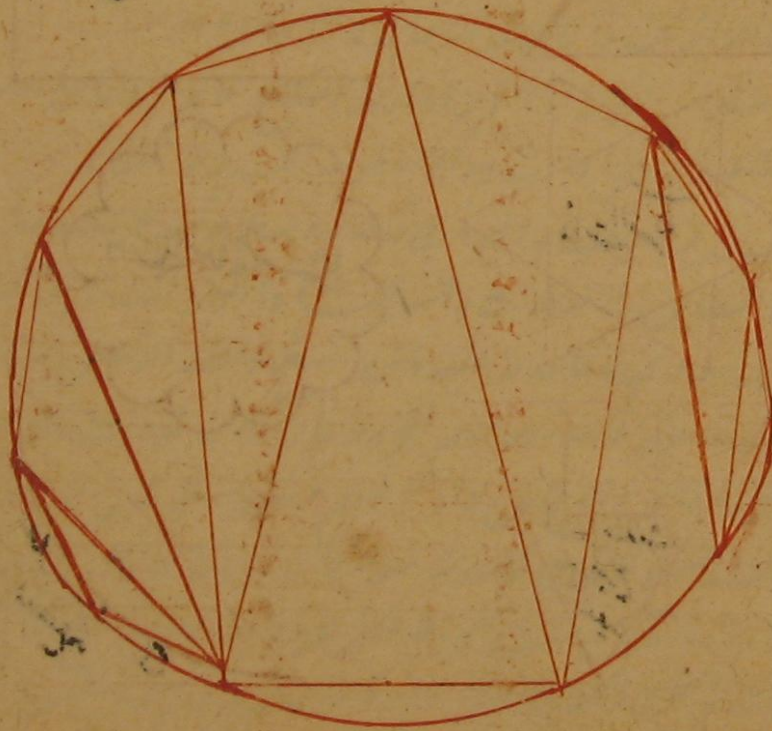
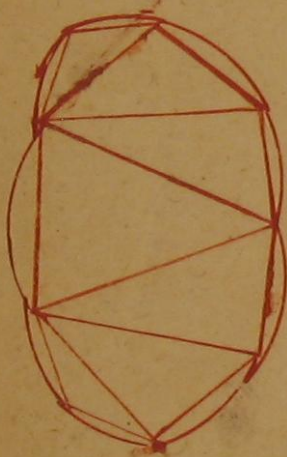
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	Sine			Difference			Arc	Sine			Difference			Arc	Sine			Difference	
	Degree	Minute	Second	Minute	Second			Degree	Minute	Second	Minute	Second			Degree	Minute	Second	Minute	Second
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1	1	2	50		48		31	30	54	8	53	35		61	52	28	38	29	59
2	2	5	38		47		32	31	47	42	52	59		62	52	58	37	29	0
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4	4	11	7		29		34	32	33	6	51	47		64	53	55	40	27	2
5	5	13	46		32		35	34	24	53	51	9		65	54	22	42	26	4
6	6	16	18		26		36	35	16	2	50	20		66	54	48	46	25	3
7	7	18	44		17		37	36	6	32	49	51		67	55	13	49	24	3
8	8	21	1	62	9		38	36	56	23	49	10		68	55	37	52	23	1
9	9	23	10	61	58		39	37	45	33	48	28		69	56	0	53	22	1
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29	29	5	19	54	41		59	51	25	48	21	52		89	59	59	27	0	33
														90	60	0	0		

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الباب الخامس في مساحة تساير السطوح المستوية
 التي لم يذكرها امتاسا مساحة السطح الذي يحيط به خط شبهة المستدير
 فبان نجعل فيه ذوا اضلاع كثيرة اما بحيث لا يبعد التفاوت
 بين السطح المحاط بالخط المستدير والسطح المحاط بالاضلاع
 والماجب يكون التقاطعات الباقية التي تحيط بكل واحد منها ضلع واحد
 من الاضلاع المعمولة وقطعة من الخط الشبيه بالمستدير قريبه
 بتقطعات مع مساحة الاضلاع يكون مساحة ترتيب

والسطح المضلع الما

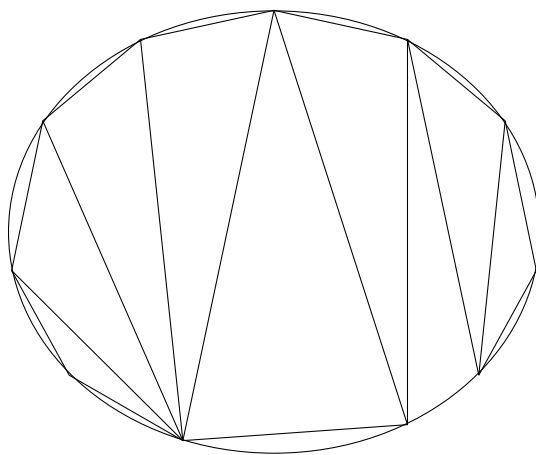
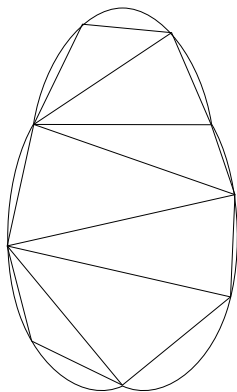


الذي الحقيقت لا يصير بينهما شيء يجمع مساحة القطر

Fifth Chapter

On the area of all planar surfaces that we have not mentioned

As for the area enclosed by a circle-like curve, [we obtain it] by constructing a polygon inside it such that we can either ignore the difference between the area of the curve and the area [of the polygon]⁵⁸ or such that the remaining portions each enclosed by one of the constructed sides and a part of the semi-circular curve are similar to [an actual circle with a negligible difference⁵⁹. Then, the sum of the areas of the portions]⁶⁰ added to the area of the polygon is approximately the area of the figure.

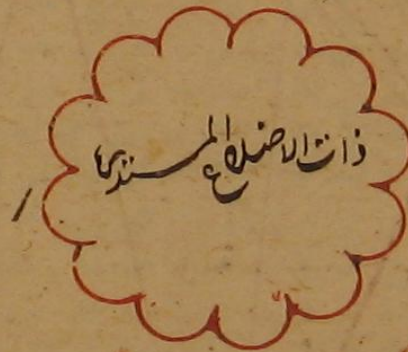


⁵⁸ Found on the left margin with a text insert sign.

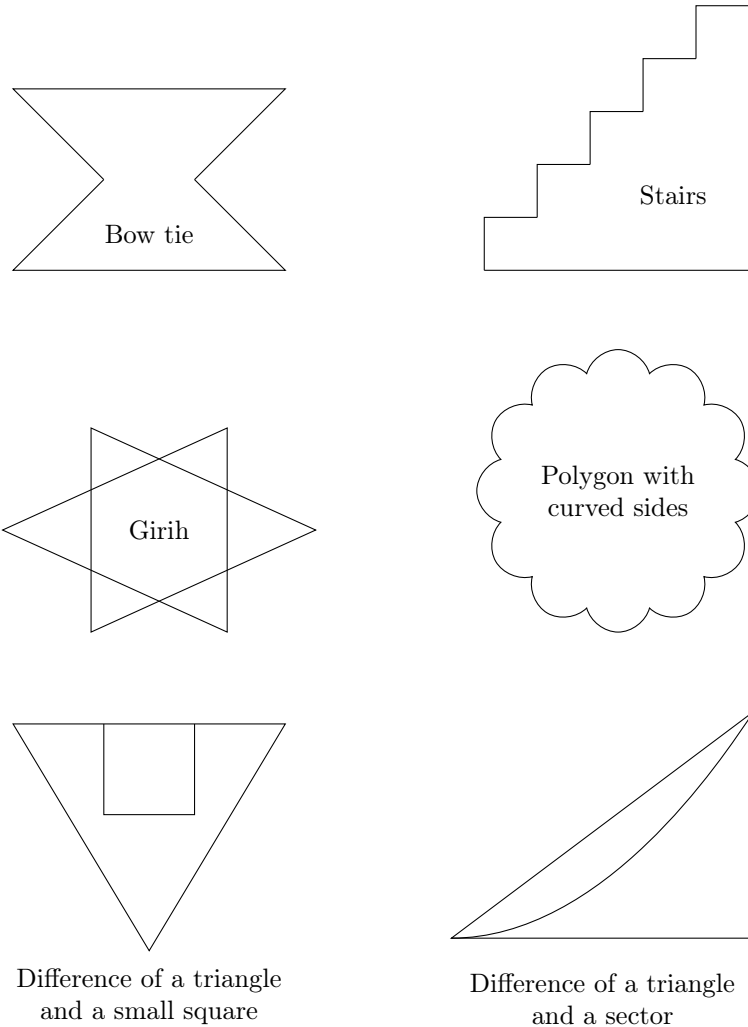
⁵⁹ This is an approximation of the area of any curve using polygons, and by looking at the figures included, one can affirm the use of triangulation to approximate any area.

⁶⁰ Found in the right margin with no text insert sign. It is found in the text in manuscript [5], for example.

وأما ما ذكرنا من السطوح المستوية كالمطبل والمدرج وذوات
 الشرفات وذوات الأضلاع المستديرة وغير ما في سهل علم من
 اطلع على ما ذكرنا بان بقطعة الى الاشكال المذكورة او يزيد فيه شيا
 الى ان يصير الى الاشكال المذكورة وبعد المساحة تنقص مساحة
 ما زاوية صور الاشكال المذكورة



As for the area of other planar surfaces, such as bow tie, stairs, girih, and polygons with curved sides, it is easy for those who looked at what we have mentioned to divide these figures or add something to them so they can become like the aforementioned figures. Then, after calculating the area, we subtract the added area. Here are the mentioned figures.



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الباب السادس في مساحة السطوح المستديرة كسطوح
الاسطوانات والمخروطات والأكروما يتعلق بها وهو مشتمل
على ستة فصول **الفصل الأول** في التعريفات الاسطوانة
المستديرة مجسم محيط به دائرتان متساويتان متوازيتان هما قاعدتاها
وسطح مستدير الوض مستقيم في الطول واصل بين قاعدتيها بحيث
إذا ادبر مستقيم واصل بين محيطي القاعدتين عليهما متوازيان مستقيمان
واصل بين مركزي القاعدتين ما بين السطح والخط الواصل بين
المركزين هو سهم الاسطوانة ويدعى محورها أيضا فان كان عمودا
على الدائرتين فالاسطوانة قائمة والا فمائلة **تعريف آخر**
للاسطوانة القائمة إذا ادبر ذو اربعة اضلاع قائم الزوايا على احد
اضلاعه فالشكل الحادث هو الاسطوانة المستديرة القائمة **المخروط**
المستدير مجسم محيط به دائرة على قاعدة و سطح مرتفع عن محيطها
على التضايق الى نقطة تسمى رأسه بحيث إذا ادبر المستقيم واصل
بين رأسه ومحيط قاعدة عليه ماس السطح والخط الواصل بين
رأسه ومركز قاعدة هو سهم المخروط فان كان عمودا على قاعدة
فالمخروط قائم والا فمائل وإذا اتوهم قطعة بسطح يكون سهمه في
ذلك السطح قائما على قاعدة سواء كان المخروط قائما او مائلا فالمثلث
الحادث فيه يسع مثلث المخروط وكل مخروط إذا فصل بسطح مواز
لقاعدته كان ذلك الفصل دائرة والسهم يمر بمركزها وينقسم إلى

Sixth Chapter

On the area of circular surfaces such as the surfaces of cylinders, cones, and spheres, and related matters

It contains six sections.

First Section. On definitions. A circular cylinder is a solid figure enclosed by two equal circles which are its bases, and a surface that has circular horizontal cross-sections and planar vertical cross-sections connecting its two bases, such that if you rotate a line connecting the circumferences of the two bases around said bases parallel to a line that connects the centers of the bases, then that line [the rotated one] would be tangent to the surface. The line connecting the centers [of the bases] is the arrow of the cylinder, and it is also called its axis. If this line is perpendicular to the bases, then the cylinder is a right cylinder, if not, then it is an oblique cylinder.

Another definition for a right cylinder: if you rotate a right quadrilateral around one of its sides then the result is a circular right cylinder. **A circular cone** is a solid enclosed by a circle which is its base, and a circular surface that ascends from its [the circle's] circumference in a narrowing fashion up to a single point which is the apex of the cone. If the line connecting the cone's apex and the circumference of its base is rotated around that circumference, then it would be tangent to the surface. The line segment connecting its apex and the center of its base is the arrow of the cone. If the arrow of a cone is perpendicular to its base then the cone is a right cone, otherwise, it is an oblique cone. If you were to imagine the cone being cut with a plane so that its arrow in that plane is perpendicular to its base, whether the cone is right or oblique, the resulting triangle is called the cone's triangle. If a cone is cut by a plane parallel to its base, then the cross section is a circle and the arrow passes through its center. The original cone is separated by the plane into

مخروط اصغر منه مشابه له ومجسم سمي بمخروط ناقص واذا ادير مثلث
 قائم الزاوية على احد ضلعي القائمه فالشكل الحادث هو المخروط
 المستدير القائم واذا ادير دوزنقه واحد على ضلعي القائم على
 المتوازيين فالشكل الحادث هو المخروط الناقص القائم وذلك لخط
 سهم ومخون وارتفاعه ايضا والمركب من مخروطين قائمين قاعدتهما
 دائرة واحد سمي بالمعين المجسم واذا افترز عن مخروط قائم معين
 مجسم يكون احد راسيه مركزا قاع المخروط فاسمى المجسم الباقي
 بقضل المخروط وهو كالمخروط ناقص افترز منه مخروط راسه مركزا قاعه
 المخروط الاول وقاعدته السطح الاعلى للمخروط الاول واذا افترز عن
 معين مجسم معين مجسم اخر يكون راسا احدهما راس الاخر فاسمى المجسم
 بقضل المعين وهو مركب من مخروطين قائمين احدهما تام والاخر ناقص
 قاعدتهما واحد افترز منه مخروط راسه راس المخروط التام وقاعدته
 السطح الاعلى من المخروط الناقص واعلم ان الاسطوانه والمخروط قد يكونان
 مضلعين قاعدتهما ذوات الاضلاع والسطح المحيط بالاسطوانه مستقيما
 وبالمخروط مثلثات **المنشور** اسطوانه قاعدتهما مثلثان متساويان
 اضلاع احدهما يوازي اضلاع الاخر **الكرة** جسم محيطه سطح مستدير
 في داخله نقطه يكون كل الخطوط الخارج عنها اليه متساويه وتلك النقطه
 مركزها والخطوط انصاف اقطارها وذلك السطح محيطها واعظم دائرة
 يقع فيها ما يمر بمركزها ولا بد منضنها واذا قطعت الكرة بسطح مستو الى

a similar smaller cone and a solid called a frustum. If a right triangle is rotated around one of its right sides, then the resulting figure is a right circular cone. If a trapezoid with one acute angle is rotated around its side that is perpendicular to its two parallel sides, then the resulting figure is a right frustum, and that line is its arrow, axis, and height. The figure resulting from combining two right cones with one circle as their shared base is called a solid rhomboid. And if a solid rhomboid is taken out of a right cone such that one of its vertices is the center of cone's base, then we call the remaining solid figure the remainder of the cone. It is similar to a frustum with a cone taken out of it such that its apex is the center of the first cone and its base is a higher surface of the first cone. If a solid rhomboid is taken out of another solid rhomboid such that two vertices of one are also vertices of the other, then we call the remaining solid figure the remainder of the rhomboid. This figure is made up of the combination of a right cone and a right frustum sharing one base with a cone taken out of it whose apex is the same as the apex of the full cone, and whose base is the higher surface of the frustum. Know that a cylinder and a cone can have sides, their bases could be polygons and the surface surrounding the cylinder becomes rectangles and the one surrounding a cone becomes triangles. **A prism** is a cylinder with bases, two equal triangles, the sides of one of the triangles are parallel to the sides of the other. **A sphere** is a solid enclosed by a round surface, and inside it there is a point such that all the lines coming out of it to the surface are equal. That point is the sphere's center, the lines are its radii, and the surface is its boundary. A great circle in the sphere is one that passes through its center, and it has to bisect the sphere. If a sphere is cut with a plane into

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قسمين فيقال لكل واحد منهما قطعة الكرة والدائرة التي حدثت فيها
 من قاعد القطعة ورأس القطعة نقطة على سطحها المستديرينساوي
 جميع الخطوط الخارجة منها الى محيط القاعدة ويقال لها قطر القطعة ايضا
 والخط الواصل بين مركز القاعدة ورأس القطعة هو ارتفاع القطعة
 وسماها ايضا قطاع الكرة هو مجموع قطعة الكرة ومحزوظ مستدير
 قائم قاعدته قاعدة القطعة ورأسه مركز الكرة ضلع الكرة
 هو ما احاط به نصفان عظيمين و سطح كروي يكون نصف قطرها
 مساو لنصف قطر الدائرتين وهو شبه اضلاع البطح الفلك
 اسطوانة مجوفة متساوي الخشن لا يكون سمكها اكثر من قطر قاعدتها
 فيكون قطر قاعدتها تجويفها اقل من نصف قطر قاعدتها او مساويا له
 سواء كان ثخنه اقل من سمكها او اكثر وما كان قطر قاعدتها التجويف
 اكثر من نصف قطر قاعدتها بحيث يكون ثخنه اقل من سمكه تسميه
 بالدفى وما كان سمكه اكثر من قطر القاعدتين مطلقا فهو الانبوبة
 وبعبارة اخرى اذا ادير سطح مستطيل حول خط خارج منه مواز
 لضلع الاقصر بعينه لا يكون اكثر من ضلعه الاطول او كان ذلك
 الخط موازيا لضلع الاطول ولا يكون ضلعه الاقصر اقل من بعده
 ولا يكون مجموعها اكثر من ضلع الاطول فالشكل كادث هو ما سميها
 الفلكه وان كان ذلك الخط موازيا لضلع الاطول ويكون ضلعه
 الاقصر اقل من بعينه ومجموعها اكثر من ضلع الاطول فالشكل كاي

two parts, then each one is called a spherical cap, and the circle that appears [from the cut] is the base of the cap. The vertex of the cap is a point on its curved surface such that all the lines connecting it to the circumference of the base are equal. It is also called the pole of the cap. The segment connecting the center of the base and the vertex of the cap is the height of the cap and its arrow. **A spherical sector** is the union of a spherical cap and a circular right cone whose base is the cap's base and whose vertex is the center of the sphere. **A spherical wedge** is what is enclosed by two semi-disks, and a curved surface with radius equal to the radii of the two disks. It looks like a slice of a watermelon. **A cylindrical shell** is a hollow cylinder with a constant width, whose height is not larger than the diameter of its base, and the diameter of its hollow portion is less than or equal to the radius of its base, whether its width is less than its height or larger. If the diameter of the base of its hollow portion is larger than the radius of its base and its width is less than its height, then it is called timbrel. And if the height is larger than the radius of the base regardless [of any other factors], then it is called a tube. In other words, if a rectangular surface is rotated around a line outside of it and parallel to its shortest side with a distance no larger than the rectangle's longer side, or if this line is parallel to the longer side at a distance less than the shorter side, where the sum [of the distance and the shortest side] is not greater than the longer side, then the resulting figure is what we call the washer. If that line is parallel to the longer side, and the shorter side is less than the distance [between the line and the rectangle], and the sum [of the distance and the shortest side] is greater than the longer side, then the resulting figure

ما سميها بالدفى وان كان مجموعها اقل منه سوار كان بعد الخط اقل من
 ضلعه الا قصر او اكثر منه فهو الانبوبة وكل سطح ادير حول خط خارج
 عنه غير مواز لضلعه الا طول ان كان مستطيلا مطلقا او مواز
 لضلعه الا قصر او لا احد اضلاع المربع ويكون بعد عنه اكثر من اعظم
 اضلاعه فالشكل الحادث نسبة بالخلقة وينسب الى سطح حادث
 فيها عن تصور قطعها بسطح يكون محورا فيه فالخلقة المربعة ما كان
 السطح الحادث فيها مربعا والمستدير ما كان دائرة وعلى هذا
 التماس والخلقة المربعة اما ان يكون احد اضلاع مربعة موازيا لمحور
 او لا ويقال للثنان بالمربعة الموربة وبعض رسم الدفى بكرة مجموعة
 متساويين الثمن افز عنها قطعتان يكون قاعدتهما متساويتان متوازيتان
 وما قلنا فهو اشبه بالدفى عن هذا **الفصل الثاني**
 فى مساحة سطح الاسطوانة اما القايمة فنضرب محيط القاعدة فى
 الخط الداىل بين محيطي القاعدتين الموازي لهما الاسطوانة و
 وهكذا يكون مساحه سطحى الداخلة والخارجة للثقله والدفى والانبوبة
 والخلقة المربعة والمستطيلة التى كان ضلعان منها موازيا لمحور
نوع اخر مخصوص بالمستدير فنضرب قطر القاعدة فى ذلك الخط
 ثم فنضرب الكاىل فى نسبة المحيط الى القطر واما المائلة فنضرب الخط
 المذكور فى محيط قطع يكون سهم قائما عليه **الفصل الثالث**
 فى مساحة سطح المخروط اما المستدير القاييم فنضرب نصف محيط

سطحى

is what we call the timbrel. If the sum is less [than the longer side], whether the distance of the line is less than or more than the shorter side, then it is a tube. Any surface rotated around a line outside of it that is not parallel to its longer side in the case of a true rectangle or parallel to its shorter side or to one of the sides of a square, where the distance from it is larger than the largest of its sides, then we call the resulting figure a ring. We call it after a surface formed by imagining a surface cross section that contains its axis. So, a square ring is a ring that has a square cross section, and a circular ring has a circular [cross section], and so on. In a square ring, one of the square's sides is either parallel to the ring's axis, or not. In the latter case, we call the ring an oblique square ring. Some people have drawn the timbrel with a hollow ball with a constant width with two portions taken out of it. The portions have equal parallel bases. However, what we described is closer to a timbrel than that.

Second Section. On the surface area of a cylinder. ⁶¹ As for a right cylinder, we multiply the circumference of the base by the segment connecting the centers of the two bases and parallel to the arrow of the cylinder. Similarly, we obtain the area of the interior and exterior surfaces of the cylindrical shell, the timbrel, the tube, and the square and rectangular rings with sides parallel to their axes of rotation.

Another way specifically for circular cylinders: multiply the diameter of the base by that line, then we multiply the result by the ratio of the circumference to the diameter. ⁶² As for the oblique cylinder, we multiply the aforementioned line by the circumference of a cross section such that the arrow of the cylinder is perpendicular to that cross section.

Third Section. On the surface area of a cone. As for a circular right cone, we multiply half the circumference

⁶¹ The surface area as defined here and later excludes the areas of the bases.

⁶² For a circular right cylinder with radius r and height h , the area is $2\pi rh$. As mentioned earlier, the areas of the bases are not considered.

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في

القاعدة في الخط الواصل بين رأسه ومحيط قاعدته ليحصل المساحة
 او ضرب نصف قطر القاعدة في ذلك الخط ثم النسبة بين القطر
 والمحيط وفي المحروط الناقص المستدير القائم ضرب نصف مجموع
 محيط الدائرتين في اقصر الخط الواصل بين المحيطين اعني الذي
 كان مع السهم في سطح واحد ليحصل المساحة او ضرب مجموع نصفي القطرين
 في ذلك الخط ثم احاصل في النسبة المذكورة وان لم يكن الخط المذكور
 معلوما وكان ارتفاعه معلوما فخذ نصف التفاضل بين قطري القاعدتين
 ورتبه مربعه على مربع ارتفاعه وناخذ جذرا الحاصل فهو مقدار الخط
 المذكور واما المستدير المائل فلم يذكر المتقدمون مساحه سطحه اذ لم يوفقوا
 الى تحصيلها سبيل ففهم في محال في معرفتها بتقريب لا يبعد عن
 الصواب وذلك ان يحصل اعظم الخطوط الخارجة من رأس المحروط
 الى محيط قاعدته واقصرها وكذا محيط قاعدته بمقياس واحد
 ثم نجري محيط قاعدته اجزاء يكون التقادرات بين كل جزء منها وبين
 وتر ذلك الجزء شيئا يسيرا بالنسبة الى المقياس ونستخرج مقادير
 الخطوط الخارجة عن رأس المحروط الى محيط قاعدته يكون البعد
 بين كل اثنين منها من محيط القاعدة بقدر جزء واحد من تلك الاجزاء
 ثم نجمع جميع مقادير تلك الخطوط ونضرب في مقدار نصف جرد واحد
 من تلك الاجزاء ليحصل المساحة ومعرفة استخراج مقادير تلك
 الخطوط المذكورة ان نعرف بعد كل خط منها عن طريق اقصر الخطوط

of the base by the line connecting its apex to the circumference of the base. The result is the area. Or, we multiply the radius of the base by that line then [by]⁶³ the ratio of the circumference to the diameter.

In a circular right frustum, we multiply half the sum of the circumferences of the two circles by the shortest segment connecting the two circumferences. Such segment would lie on the same plane as the arrow. The result is the area. Or, we multiply the sum of the two radii by that segment, then by the aforementioned ratio. If the mentioned segment is unknown and the height is known, then we take half the difference between the diameters of the bases and add its square to the square of the height, then we take the square root of the result to get the length of the mentioned segment. As for an oblique circular cone, my predecessors have not mentioned a way to calculate its surface area because there was⁶⁴ no way to obtain it.⁶⁵ So, we use a trick to find it with an approximation that is not far from the correct answer. We obtain the longest segment coming out of the apex of the cone to the circumference of its base, as well as the shortest such segment. We also calculate the circumference of its base in the same units. Then, we divide the circumference of the base into parts such that the differences with their respective chords are small in terms of the used units. We obtain the measures of the segments connecting the apex of the cone to the circumference of its base with a distance between each pair, as much as one part of those parts. Then, we add the lengths of those segments and multiply the sum by half the length of one of the parts to get the area⁶⁶. To know the lengths of the aforementioned segments, we find the distance between each one of them and the end of the shortest segment

⁶³ Found in the left margin with an insert sign.

⁶⁴ Could also be understood as “there is”.

⁶⁵ Al-Kāshī is recognizing that there is no formula for the surface area of an oblique cone.

⁶⁶ Al-Kāshī is using the sum of small subdivisions of the circumference of the base times heights to approximate the surface area of an oblique cone. This is an integral approximation.

من اجزاء محيط القاعدة كم كان بماية محيط القاعدة ثلثية وستون و
 يعرف كل واحد من حيزه وسهمه ثم نقسم نصف المحيط على نسبة
 المحيط الى القطرين فما خرج فهو نصف قطر القاعدة ضربا به كل
 واحد من الجيب والسهم المذكور من خط ونسج حاصل ضرب الجيب
 بالمخفوف الاول وحاصل ضرب السهم بالمخفوف الثاني ثم نضرب مجموع
 الضلعين الاطول والاقصر في تفاضلهما ونقسم الحاصل على قطر
 قاعدة فما خرج نأخذ التفاضل بينه وبين قطر القاعدة ويضاف
 فهو بعد موقع العمود الخارج عن رأس المخروط على سطح قاعدة
 عن طرف اقصر الاضلاع ونسميه بالمخفوف الثالث ننقص مربع
 عن مربع اقصر الاضلاع بنسج مربع العمود ثم نجعل بين مخفوفى الثاني
 والثالث ونسميه بالمخفوف الرابع ونجعل مربعه مع مربع العمود و
 المخفوف الاول ونأخذ جذر المجموع فهو الخط الملتصق وأما مساحة
 سطح المخروط المضلع هي مجموع مساحة المثلثات التي يحيط به
الفصل الرابع في مساحة سطح الكرة واستخراج قطرها أما
 المساحة فنضرب القطر في محيط اعظم دائرة يقع فيها يحصل المساحة
نوع آخر نضرب مربع القطر في نسبة المحيط الى القطر ليحصل المساحة
 وهو اربعة امثال اعظم دائرة يقع فيها واما سطح اسطوانة مستديرة
 قائمة سوى القاعدتين يكون كل واحد من سمكها وقطر قاعدتها
 مساويا لقطرها ويساوى ايضا سطح اسطوانة مع القاعدتين

of the circumference of the base. Since there are three hundred sixty parts in the circumference of the base, and we know each of their sine and versine, we divide half the circumference by the ratio of the circumference to the diameter. What is obtained is the radius of the base, which we multiply by each of the aforementioned sines and versines, reduced [by a factor of 60]. We call the result from multiplying by the sine the first saved value, and the result from multiplying by the versine the second saved value. Then we multiply the sum of the longest side and the shortest side by their difference, and divide the result by the diameter of the base. We take the difference between the result and the diameter of the base, and halve it. What we get is the distance between the perpendicular coming out of the cone's apex on the surface of its base to the side of the shortest line. We call this value the third saved value. We subtract its square from the square of the shortest side, and what remains is the square of the height. Then we add the second and the third saved values and call the result the fourth saved value. We add its square to the squares of the height and the first saved value, and take the square root of this sum to get the wanted segment. As for the area of an edged cone, it is the sum of the areas of the triangles that surround it.

Fourth Section. On the surface area of a sphere and obtaining its diameter. As for the area, we multiply the diameter by the circumference of a great circle that contains it. The result is the area.

Another method. We multiply the square of the diameter by the ratio of the circumference to the diameter to get the area. The area is four times that of a great circle, and is equal to the surface of a right circular cylinder without the two bases, such that the height of the cylinder and the diameter of its base are equal to the diameter of the sphere. It is also equal to the surface of a cylinder with the two bases

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فجار برکار
ع

يكون سمكها مساويا لنصف قطرها وقطر قاعدتها مساويا لنقطتها **والتاسعة**
 استخراج قطرها فبان نجعل نقطه من سطحها قطبا ونضع عليها احدى رجلي
 الفجار ونرسم بالرجل الاخرى محيط دائرة على سطح الكرة ونضع بهذا
 الفتح على خط مستقيم ونمسح بين رجلي الفجار ونسميه بالمقدار الاول
 ثم نقسم محيط تلك الدائرة ستة اقسام متساويات بالزجار ونحصل
 مقدار هذا الفتح بتلك الاجزاء ايضا وينقص مربعه من مربع المقدار ^{الاول}
 وناخذ جذر الباقي فهو ارتفاع قطعه يكون سطح الدائرة المرسومة قاعدتها
 فنقسم عليه مربع المقدار الاول فما خرج فهو قطر الكرة **نوع لفر** نرسم على
 الكرة دائرة كيف اتفقت ونحفظ فتح الزجار ونسميه بالفتح الاول
 ثم نقسم تلك الدائرة اقسامه اقسام وناخذ منها ثلثه اقسام واما نسبها
 اربعة اقسام وناخذ منها قسمين بزجار اخر ونسميه بالفتح الثاني ثم نرسم
 على سطح متوجع خطا مستقيما ونضع عليه بالفتح الثاني نقطتين ونرسم على
 كل واحد منهما ببعده الفتح الاول دائرة فالدائرتان يتقاطعان البته ثم
 نرسم على احد تقاطع ما بين الدائرتين دائرة بالفتح الاول ايضا فيستاق
 مع كل واحد من الاولين على نقطتين نصل بينهما خطا وكذا بين الآخرين
 فيستاق هذا ان الخطان البته فمن هذا التقاطع الى كل واحد من النقطتين
 الموضوعتين اولا هو نصف قطر الكرة **كذا**

such that its height is equal to its radius [that of the sphere] and the diameter of its base is equal to its diameter [that of the sphere]. As for obtaining its diameter, we fix a point on its surface as the pole and put one of the compass⁶⁷ legs on that point and draw with the other a circumference of a circle on the sphere. We then put this gap [the compass] on a straight line and measure between the legs of the compass. We call this value the first value. Then, we divide the circumference of this circle into six equal parts using the compass. We obtain the measure of that gap using the same units used previously and subtract its square from the square of the first value. We take the root of the remainder to get the height of a cap whose base is the surface of the drawn circle. We divide the square of the first value by that height. What results is the diameter of the ball.

Another method. We draw an arbitrary circle on the sphere. We save the opening of the compass and call it the first gap. Then, we divide that circle either into six parts and take three or divide the circle into four parts and take two using another compass. We call that gap the second gap. Then, we draw a straight line on a plane surface, and we put two points on it using the second gap. We draw a circle on each of them using the first gap. The two circles will definitely intersect. Then, we draw another circle on one of the circles' intersection points using the first gap as well. It will intersect with the first two circles in two points each. We draw a line connecting them, and we do the same for the others. These two lines will definitely intersect. [The distance] between this intersection and each of the two points determined initially is the radius of the sphere, like this:

⁶⁷ Compass is written in Arabic and Persian on the left margin as well.

الفصل الخامس في مساحة سطح المستدير لقطعة الكرة و

واستخراج ابعادها بعضها من بعض اما المساحة فنضرب الخط الواصل بين رأس القطعة ومحيط قاعدتها في نسبة المحيط الى القطر ثم في الحاصل يحصل مساحة القطعة وهي يساوي لدائرة يكون نصف الموضوفا بقدر الخط المذكور **نوع اخر** نضرب ارتفاع القطعة في محيط اعظم دائرة يقع في تلك الكرة يحصل المساحة واما استخراج ابعادها اذا كان نصف قطر قاعدتها وارتفاعها معلومين بنجمع مربعيهما فمأخذ جذر المجموع هو الخط الواصل بين رأس القطعة ومحيط قاعدتها ولو تقسم مربع نصف قطر قاعدتها على ارتفاعها فخرج نريد على ارتفاعها لكان المجموع قطر الكرة نضرب في نسبة المحيط الى القطر اعني في ٩ ح ٣ ح ٤ يحصل محيط اعظم دائرة تقع فيها

الفصل السادس في مساحة سطح المستدير لضعف الكرة نضرب

قطر الكرة في اعظم الميل بين الدائرتين المحيطتين به يحصل المحيط

الباب السابع في مساحة الاجسام

يشتمل على ثمانية فصول **الفصل الاول** في مساحة الاسطوانة نضرب

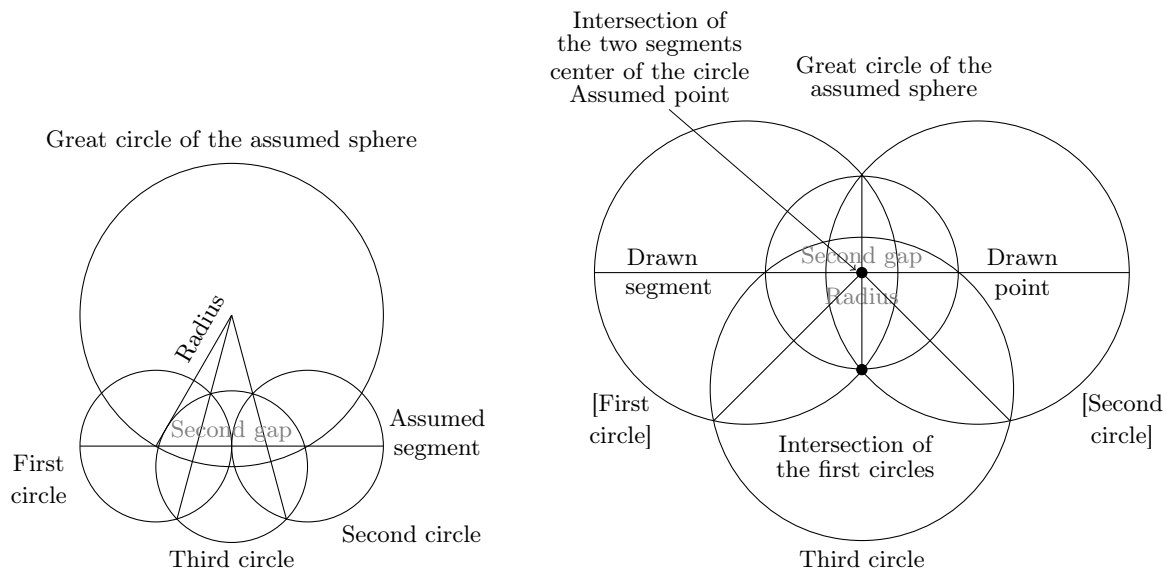
مساحة احد قاعدتها في العمود الواقع على سطحها اما داخل الاسطوانة

او خارجها وهو في الاسطوانة القائمة سهمها واما استخراج عمودها

في المائل فبان نضرب جيب زاوية ميلها في الخط الواصل بين

محيطي القاعدتين الموارس والمساوي سهمها من خط يحصل عموده





Fifth Section. On the area of the curved surface of a sphere's cap and obtaining some of its dimensions from others. As for the area, we multiply the square of the length of the segment connecting the cap's vertex and the circumference of its base by the ratio of the circumference to the diameter. This area is equal to the area of a circle with radius the same length as the mentioned segment.

Another method. We multiply the height of the cap by the circumference of the sphere's great circle to get the area. As for obtaining its dimensions, if the radius of the cap's base and its height are known, then we add their squares and take the square root of the sum to get the segment connecting the cap's vertex to the circumference of its base. If we divide the square of the base's radius by the height and add the result to the height, the result then is the diameter of the sphere. We multiply it by the ratio of the circumference to the diameter i.e., by 3;8 : 29 : 44 to get the circumference of the great circle.

Sixth Section. On the area of the curved surface of a spherical wedge. We multiply the diameter of the sphere by the biggest arc between the two circles enclosing the wedge to get the wanted value.

Seventh Chapter On the volume of solids

It contains eight sections:

First Section. On the volume of the cylinder. We multiply the area of one of its bases by the line segment perpendicular to its [base's] surface, whether inside the cylinder or outside of it. In a right cylinder, this line is the the height of the cylinder. As for obtaining the height in an oblique cylinder, we multiply the sine of the angle of its slant by the segment connecting the circumferences of the two bases, where the segment is parallel and equal to the arrow reduced by a factor of 60. The result is its height.

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الفصل الثاني في مساحة المخروط واستخراج عموده أما المساحة
فنضرب ثلث مساحة قاعدته في العمود الخارج عن رأس المخروط على
سطح قاعدته إذا كان أو خارجاً **نوع آخر** مخصوص بالتأثير المستدير
فنضرب ثلث العمود الخارج من مركز قاعدته الواقع على ضلع من أضلاع
اس على خط واصل بين رأسه ومحيط قاعدته في سطح المستدير ليحصل
المساحة وأما استخراج العمود الخارج عن رأس المخروط على سطح
قاعدته إذا كان قطر قاعدته والخط الواصل عن رأس المخروط و
محيط قاعدته معلومان في التأثير المستدير أو الخطان الأطول والأصغر
في المائل المستدير وهما مع قطر القاع يكون أضلاع مثلثة فيستخرج
العمود عن أضلاع مثلثة كما سبق في مساحة المثلث وإن كان
المخروط مضلعاً قائماً ويكون أضلاع قاعدته بحيث يمكن أن يحيط بها
دائرة تماس جميع زوايا فينقص مربع نصف قطر تلك الدائرة عن
مربع الخط الواصل بين رأس المخروط وواحد زوايا القاعدته أو
يمكن أن يحيط به دائرة تماس أضلاعها فينقص مربع نصف قطرها
عن مربع الخط الواصل بين رأس المخروط وواحد نقط التماس
فما بقي فهو مربع العمود وإن كان المخروط مضلعاً مائلاً ويكون أضلاع
قاعدته متساويات ويكون السطح الموهوم المار بسهم التأثير على
قاعدته ماراً بأحد زوايا قاعدته ومقتصف أحد أضلاع
فيما كان عدد أضلاعه فرداً وأما بالزوايتين المتقابلتين أو

Second Section. On the volume of a cone and obtaining its height. As for the volume, we multiply one third of the area of its base by the height coming out of the cone's apex to the surface of the base, whether within the cone or outside of it.

Another method specific to a circular right cone: we multiply one third of the height coming out of the center of its base on one of its sides, i.e., on a line connecting its apex to the circumference of its base, by the area of its circular surface to get the volume. As for obtaining the height coming out of a cone's apex onto its base, in a right circular cone, if the diameter of its base and the line connecting the apex to the circumference of the base are known, and in a slanted circular cone, if both the longest and shortest segments are known which, with the diameter of the base, form the sides of its triangle, then we obtain the height from the sides of the triangle as we did in the case of the area of a triangle. If the cone is a right polygonal cone so that a circle includes all the vertices of its base, then we subtract the square of this circle's radius from the square of the segment connecting the cone's apex to a vertex on the base. Or, if it is possible to have a circle tangent to the sides of the base, then we subtract the square of its radius from the square of the segment connecting the cone's apex to one of the tangent points [of the circle and sides]. What remains is the square of the height. The case of an oblique polygonal cone with equal sides of the base: if the imagined plane that passes through its arrow which is perpendicular to its base either passes through a vertex of the base, and the middle of one of cone's sides in the case the cone has an odd number of sides, or if it passes through either two opposite vertices, or

بمنتصفي الضلعين المتقابلين فيما كان عدد اضلاعه زوجا او يقطع الضلعين
المتقابلين على غير تقاطع المنتصف فيحدث فيه من ذلك السطح مثلث
يكون قاعدته فيما كان اضلاعه قاعدته فردا بقدر مجموع نصف قطري
الدائرة الداخلة والخارجة واحد ساقية بقدر الخط الواصل بين رأسه و
الزاوية والاخر بقدر الخط الواصل بين رأسه ومنتصف الضلع فيستخرج
منها العمود كما سبق في مسافة المثلث واما فيما كان اضلاعه قاعدته زوجا
فان كان السطح مارا بالزاويتين منها فيكون قاعدته مثلث المحزوط
قطر الدائرة الخارجة بزوايا القاعدتين واحد ساقية اللاتول الواصل بين
رأسه ومحيط قاعدته والاخر الاقصر الواصل بهما وان كان السطح مارا
بمنتصفي الضلعين فيكون القاعدتين قطر الدائرة الداخلة والضلعان
الاخران هما اطول الخطوط الواصلة بين رأسه ومنتصف اضلاعه الثابتين
واقصر ما يستخرج منها العمود وان كان قاطعا للضلعين على غير تقاطع
المنتصف نزله مربع بعد التقاطع عن منتصف الضلع على مربع نصف
قطر الدائرة الداخلة وناخذ جذر المجموع وضعفه فهو قاعدته مثلث المحزوط
والخطان الواصلان بين رأس المحزوط وطرفي القاعدتين هما ساقاه
يستخرج منها العمود **نوع اخر** اعلم منه ان كان سهم معلوما وكذا زاوية
الميل منخطا فما حصل فهو العمود وكذا الحكم في كل خط وصل عن
رأس المحزوط ومحيط دائرته اذا كان مقدار زاوية ميل ذلك الخط
معلوما وهذا شأن جميع المحزومات واما استخراج العمود الخارج

the midpoints of two opposite sides if the cone has an even number of sides, or if it cuts two opposite sides but not through their midpoints, then that plane forms a triangle in the cone. The base of this triangle, in the case of the cone having odd sides, is equal to the sum of the radii of the inner and outer circles, and one of triangle's sides would be equal to the segment connecting the [cone's] apex and the vertex [of the side]. The other side would be equal to the segment connecting the apex to the middle of the side. We obtain the height from these values like previously done in the case of the area of a triangle. If the number of the sides in the cone's base is even, and if the plane passes through the two angles of it, then the base of the cone's triangle is the diameter of the circumcircle of the base, and one of its sides is the longest segment connecting the cone's apex to the circumference of its base, and the other side is the shortest segment connecting them. And, if the plane passes through the midpoints of the two [opposite] sides, then the base of the triangle is the diameter of the incircle, and the other two sides are the longest and shortest segments connecting the apex to the midpoints of the base's sides. We then obtain the height. And if the plane cuts two sides but not at their midpoints, then we take the square of the distance between the point of intersection and the middle of the side and add it to the square of the radius of the incircle. We take the square root of the sum and double it, then the result is the base of the cone's triangle, and the two segments connecting the cone's apex to both ends of the base are the legs of the triangle. From this, we then obtain the height.

A more general method. If the arrow and the slant angle reduced are known, then the product [of the arrow by the sine of the complement of the slant angle reduced]⁶⁸ is the height. This method can be used to obtain any segment connecting the apex of the cone and the circumference of its base if the slant angle of the segment is known. The method is also inclusive of all cones. As for obtaining the height coming out

⁶⁸ This part is missing but can be found in manuscripts [3], and [5].

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عن مركز القاعدتين على خط واصل بين رأس المخروط ومحيط قاعدته فننصف
 مجموع سهم المخروط وننصف قطر قاعدته في تقاطعها ونقسم الكااصل على
 الخط المذكور فما خرج منقصة عن ذلك الخط ثم ننقص مربع نصف الباع
 عن مربع نصف قطر القاعدتين فما بقى فاذ جذره فهو الخط **المطلوب**
الثالث في مسافة المخروط ان قص **أما** المستدير فنضرب قطر
 قاعدته في العمود الواقع بين السطحين ونقسم الكااصل على الثناوت
 بين قطري القاعدتين والسطح الاعلى الموازي لها فما خرج فهو عمود المخروط
 التام ننقص منه العمود الاول فما بقى فهو عمود المخروط الصغير ثم نجمع
 المخروطين وننقص الاقل من الاكثر ليعتد مسافة المخروط التام
وأما المضلع فان كان اضلاع قاعدته بحيث يمكن ان يحيط بها
 دائرة تماس جميع زواياها او محيط بدائرة تماس جميع انصاف
 اضلاعها فنعمل باحد قطري الدائرة او الخارج لكل واحد من السطحين
 ما علمنا في المستدير بقطري القاعدتين وان لم يكن فيه العمود معلوما
 وكان المخروط قائما واعظم الخطوط الواصلة بين محيطي القاعدتين
 اعنى الواصل بين الزاويتين منهما معلوما فنأخذ فضل قطر الدائرة
 الخارجة للقاعدتين على الخارج ايضا للسطح الاعلى وننقص مربع نصف
 الثناضل عن مربع الخط المذكور المعلوم فما بقى فهو مربع العمود وان
 كان اصغر الخطوط الواصلة بين المحيطين معلوما اعنى الواصل بين
 الضلعين منهما القائم عليهما فنعمل بقطر الدائرة الداخلة منهما ما علمنا

of the center of the base on a line connecting the apex of the cone and the circumference of its base, we multiply the sum of the arrow and the radius of the base by their difference and divide the result by the mentioned segment. We then subtract the result from that same segment. Then, we subtract the square of half of the remainder from the square of the radius of the base. We then take the square root of the result, which is the wanted value.

Third Section. On the volume of a frustum. As for a circular frustum, we multiply the diameter of its base by the height between the two surfaces, and we divide the result by the difference between the diameters of the base and the upper surface parallel to it. The result is the height of the full cone. We subtract from it the first height. Then, what remains is the height of the small cone. We then measure [the volumes] of both cones and subtract the smaller from the bigger, the remainder would then be the volume of the frustum. As for a polygonal frustum, if it is possible to have a circle that passes through all the vertices of the base or a circle that is tangent to all its sides at the midpoints, then we use the diameter of the inner or the outer circle of each of the two surfaces the way we used the diameter of the two bases in the circular frustum.

If the height is unknown in a right frustum, and the longest segment connecting the circumferences of its two bases, i.e., between two of their vertices, is known, then we take the difference between the diameter of the circumcircle of the base and the circumcircle of the upper surface and subtract the square of half of this difference from the square of the aforementioned known segment. The remainder is the square of the height. If the shortest segment connecting the two circumferences is known, i.e., connecting two of their sides and perpendicular to them, then we use the diameter of the incircle the way we used the diameter

هناك بالخارج **نوع آخر** وان كان زاوية ميل سهم المحزوط عن النيام
 معلومة فنضرب مقدار السهم في جيب تمام تلك الزاوية منحنى يحصل
 مقدار العمود وهذا مثل للمحزوط المائل ايضا **الفصل الرابع**
 في مساحة فضل المحزوط ومساحة فضل المعين المجسم اما مساحة
 فضل المحزوط فنضرب ثلث العمود الخارج عن مركز قاعدته على ضلع
 من اضلاعه في السطح المستدير للمحزوط الناقص ليحصل المساحة و
 اما مساحة فضل المعين المجسم فنضرب ثلث العمود الخارج من
 رأس المحزوط التام الواقع على ضلع من اضلاع المحزوط الناقص فبها
 كان او داخلا في السطح المستدير الواقع بين القاعدتين المشتركة وبين
 السطح الاعلى للمحزوط الناقص ليحصل المساحة **الفصل الخامس**
 في مساحة الكرة فنضرب نصف قطرها في ثلث مساحة سطحها المحيط
 بها ليحصل المساحة **نوع آخر** فنضرب ثلثي قطرها في مساحة اعظم
 دائرة يقع فيها **نوع آخر** مكعب القطر وناخذ منه احد عشر جزءا من احدى
 وعشرين بحسب المشهور واما بحسبنا فنضرب مكعب القطر في
 $\frac{1}{4}$ كما نذكر رابع وهو سدس نسبة المحيط الى القطر يحصل المساحة
نوع آخر فنضرب سدس مكعب القطر في نسبة المحيط الى القطر
نوع آخر فنضرب ثلث مكعب القطر في نسبة مساحة الدائرة الى مربع
 القطر التي هي 7 مكرر كما سبق في الباب الرابع اعلم ان الكرة
 تساوي الاسطوانة قاعدتها يساوي اعظم دائرة يقع في الكرة

of the circumcircle before.

Another method. If the slant angle of the arrow of a cone is known, then we multiply the length of the arrow by the sine of the complement of that angle reduced. The result is the length of the height. This method is inclusive of slanted cones as well.

Fourth Section. On the volume of the remainder of a cone and the volume of the remainder of a solid rhomboid. As for the volume of the remainder of the cone, we multiply one third of the height coming out of the center of the base on one of its sides by the curved surface of the frustum to get the volume. As for the volume of the remainder of a solid rhomboid, we multiply one third of the height coming out of the apex of the full cone on one of the sides of the frustum, whether within the solid or outside of it, by the curved surface that lies between the shared base and the upper surface of the frustum to get the volume.

Fifth Section. On the volume of a sphere. We multiply its radius by one third of the surface area that encloses it to get the volume.

Another method. We multiply two thirds of its diameter by the area of its great circle.

Another method. We cube the diameter and we take eleven parts out of twenty-one parts of it, following the common calculation, or according to our calculations we multiply the cube of the diameter by 0;31 : 24 : 57 : 20, which is one sixth of the ratio of the circumference to the diameter.

Another method. We multiply one sixth of the cube of the diameter by the ratio of the circumference to the diameter.

Another method. We multiply two thirds of the cube of the diameter by the ratio of the area of the circle to the square of the diameter, which is 0;47 : 7 : 26 as mentioned in the fourth section.

Note that [the volume of] a sphere is equal to a cylinder whose base equals the sphere's great circle,

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وارتفاعها بقدر ثلثي قطر الكرة وايضا يساوي لاربع مخروحات
قاعدة كل واحد منها مساوية لاعظم دائرة يقع في تلك الكرة
وارتفاعها مساو لنصف قطر تلك الكرة **الفصل السادس**
في مساحة قطاع الكرة وقطعتها فنضرب نصف قطر الكرة في ثلث
مساحة سطح الكروي يحصل مساحة القطاع ثم ينقص ارتفاع القطعة
عن نصف قطر الكرة ونضرب ثلث الباقي في سطح قاعدة القطعة
يحصل مساحة مخروط القطاع ينقصه عن مساحة القطاع الذي
هو اقل من نصف الكرة او زايده عليها ان كان اكثر فالباقي
او الحاصل هو مساحة القطعة **الفصل السابع** في مساحة الاجسام
المتساويات اضلاع القواعد يمكن ان يحيط بها محيط كرة تماس
رؤساها ويمكن ان يحيط كل واحد منها بكرة تماس مراكز قواعد
او بكرة تتن موازيتين تماس احداهما بعض قواعد الجسم والاخرى
تماس بواقيها وكل واحد منها كجسم عن مخروحات مضلعات اما
متساويات القواعد والارتفاعات او مختلفة القواعد والارتفاعات
فيكون رؤسها متحدة عند مركز الجسم وهي سبعة مجسمات اما الاول
فهو ذواربع قواعد مثلثات متساويات في الكرة وهو مجسم يحيط به
اربعة مثلثات متساويات الاضلاع وهو مخروط بمثلث القاعدة
فكان المؤلف عن اربع مخروحات قواعد ما قواعد ورؤسها مركزه و
العمل فيه ان نربع قطر الكرة المحيط به ونأخذ جذر ثلثيه وكذا جذر نصف

and whose height equals two thirds the diameter of the sphere. It also equals four cones, the base for each one equals the great circle of the sphere, and the height of each of them equals the radius of the sphere.

Sixth Section. On the volume of spherical sectors. As for sectors, we multiply the radius of the sphere by one third of the area of its spherical cap to get the volume of a sector. Then, we subtract the height of the cap from the sphere's radius and multiply one third of the remainder by the surface area of the cap's base to get the volume of the sector's cone. We subtract it from the volume of the sector if the cap is less than half the sphere and add it to the sector if the cap is more than half the sphere. The sum or the remainder is then the volume of the cap.

Seventh Section. On the volume of solids with equal sides and bases. It is possible to construct a circumsphere to such solids such that the sphere is tangent to all vertices of the solid. It is also possible to either construct a sphere that is tangent at the centers of its bases or to construct two parallel⁶⁹ spheres such that one of them is tangent to some of the solid's bases and the other one is tangent to the rest. Each one of these solids is like a union of polygonal cones, either with equal bases and heights or with different bases and heights, such that their vertices are connected at the center of the solid. And there are seven such solids.

The first one is the solid with four equal triangle bases, fitting in a sphere. It is a solid enclosed by four equilateral triangles, and it is a cone with a triangle base [Regular tetrahedron]. The solid looks like it is formed out of four cones whose bases are the bases of the solid and whose apexes are the solid's center. The computation in this case is to take the square root of two thirds of the square of the diameter of the circumsphere surrounding it, as well as the square root of half

⁶⁹ It is said parallel but we believe it is meant to be tangent.

مربع القطر فالاول ضلع القاعدة والثاني عمود مثلث القاعدة نضرب
 احدهما في نصف الاخر يحصل مساحة سطح احدى قواعده نضرب في تسعي
 قطر تلك الدائرة يحصل المساحة **نوع اخر** نضرب قطر الكرة تارة في
 مدح محيطها ما قامه يحصل ضلعها وتارة في مدحها ما قامه
 يحصل عمود المثلث والباقي كما سبق **نوع اخر** نأخذ جذر تسع مربع القطر
 ونضرب في جذر سدس مربع القطر فاحصل نضرب في ثلث القطر يحصل
 المساحة وان كان الضلع معلوما وقطر الكرة وارتفاع المجسم مجهولين
 نربع الضلع ونأخذ جذر ثلثيه فهو ارتفاع المجسم يساوي ثلثي قطر الكرة و
 تزيد نصف الارتفاع عليه يحصل قطر الكرة **نوع اخر** نضرب الضلع
 في محيطها ما قامه يحصل ارتفاع المجسم وهو ثلثي قطر الكرة
 واما الثاني فهو دوشماني قواعده مثلثات متساويات الاضلاع
 في الكرة والسعمل فيه ان نضرب قطر الكرة التي يحيط به في نصف القطر
 ثم الحاصل في ثلث القطر او نضرب مربع القطر في سدس القطر فاحصل
 هو المساحة **نوع اخر** نضرب القطر في ما قامه له
 ما قامه يحصل المساحة **نوع اخر** وان كان ضلع قاعدة معلوما
 وقطر الكرة المحيط به مجهولا فنضعف مربع الضلع ونأخذ جذره فهو قطر
 الكرة **نوع اخر** نضرب الضلع في امدها ما قامه يحصل القطر
 ثم نضرب مربع الضلع في ثلث القطر يحصل المساحة واما الثالث
 فهو المكعب الذي في الكرة والسعمل فيه ان نأخذ ثلث مربع قطرها و

يحصل

the square of the diameter. The first value is the side of the base and the second value is the height of the base's triangle. We multiply one of these values by half of the other to get the area of one of the bases. We multiply it by two-ninths of the diameter of that sphere to get the volume.

Another method. We multiply the radius of that sphere by 0;48 : 59 : 23 : 15 : 41 fifth to get the side [of the base] and by 0;42 : 25 : 35 : 03 : fifth to get the height of the triangle, and the rest is like what is mentioned before.

Another method. We take the square root of two-ninths of the square of the diameter, and we multiply it by the square root of one sixth of the square of the diameter. We then multiply the result by one third of the diameter to get the volume. If the side is known, and the diameter of the sphere and the solid's height are unknown, we square the side and take the square root of two thirds of the square of the side to get the height of the solid, which in turn equals two thirds of the sphere's diameter. By adding to it half of the height we get the diameter of the sphere. we take the square root of two thirds of the square of the side

Another method. We multiply the side by 0;48 : 59 : 23 : 15 : 41 fifth to get the height of the solid, which is two thirds of the sphere's diameter.

The second solid is the one with eight equilateral triangle bases [regular octahedron], fitting in a sphere. The way to calculate it is to multiply the diameter of the circumsphere by the radius, then by one third of the diameter, or multiply the square of the diameter by one sixth of the diameter to get the volume.

Another method. We multiply the diameter by 0;42 : 25 : 35 : 03 : 53 fifth to get the volume.

Another method. If a side of its base is known and the diameter of the circumsphere is unknown, we double the square of the side, then take its square root to get the diameter of the sphere.

Another method. We multiply the side by 1;24 : 51 : 10 : 07 : 46 fifth to get the diameter. Then, we multiply the square of the side by one third of the diameter to get the volume.

The third solid is the cube that fits in a sphere. The way to calculate it is to take one third of the square of the sphere's diameter and

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يُحصل جذر فهو ضلع المكعب يحصل منه مساحة بان ضرب في نفسه
ثم ضرب في الحاصل **نوع آخر** ضرب قطر الكرة في ٢ له كذا لقطر
الكرة حاصل ضلعها ولو قسم الضلع عليه يحصل لقطر وظاهر ان
قطر الكرة الداخلة فيه يساوي ضلعه والمكعب اسطوانة مربعة القاعدة
ارتفاعها يساوي ضلع قاعدتها قامت الاربع فهو ذو عشرين قاعدة
ثلثات متساويات الاضلاع في الكرة والعمل فيه ان نربع قطر تلك
الكرة وناخذ نصف عشرة وينقص جذر عن نصف قطر الكرة
فما بقى يحفظه ويزيد مربعه على خمس مربع القطر وناخذ جذر المجموع
فهو ضلع قاعدة المجسم **نوع آخر** ناخذ خمس مربع قطر الكرة ونضرب
جذر في اسالب ١٢ مد فالحاصل هو ضلع قاعدة المجسم
نوع آخر ضرب القطر في ٢ لالب لرند ١٢ فاحسبه وهو وتر لنصف
قوس يكون سهمها اربعة اخماس القطر على ان النقط واحد يحصل ضلع
القاعدة يحصل منه مساحة سطح القاعدة ونضربها في عشرين دائري
ليحصل مساحة جميع سطح المجسم ثم ننقص ثلث مربع الضلع عن ربع
مربع القطر وناخذ جذر الباقي فهو نصف قطر كرة يحيط الشكل بها
اعني العمود الخارج عن مركز المجسم على سطح القاعدة **نوع آخر** ضرب
قطر الكرة في ١٢ لالب ما هو حاصله يحصل نصف قطر الكرة
الداخلة ثم نضرب ثلث ذلك العمود في جميع سطح المجسم فالحاصل هو
مساحة المجسم وان كان ضلع ثلث القاعدة معلوما وقطر الكرة

obtain its square root to get the side of the cube. We obtain the volume by multiplying it by itself then multiply it [the side again] by the result.

Another method. We multiply the diameter of the sphere by 0;34 : 38 : 27 : 39 : 29 fifth to get cube's side. If we divide the side by this number we get the diameter. It is obvious that the diameter of its incircle is equal to its side. And the cube is a cylinder with a square base, its height equals the side of its base.

The fourth solid is the one with twenty equilateral triangle bases [regular icosahedron], fitting in a sphere. The way to calculate it is to square the diameter of the sphere, then take half of its tenth and subtract its square root from the radius of the sphere. We save the remainder and add its square to one fifth of the square of the diameter, then we take the square root of the sum to get the side of the solid's base.

Another method. We take one fifth of the square of the sphere's diameter and multiply its square root by 1;10 : 32 : 03 : 13 : 44 fifth. The result is the side of the solid's base.

Another method. We multiply the diameter by 0;31 : 32 : 37 : 54 : 13 fifth, which is a chord for half of an arch with arrow four-fifths the diameter assuming the diameter is one. By obtaining the side of the base, we can obtain from it the area of the surface of the base which we always multiply by twenty to get the surface area of the whole solid. Then, we subtract one third of the square of the side from one fourth of the square of the diameter, and take the square root of the remainder to get the radius of a sphere surrounded by the solid, i.e., the height coming out of the center of the solid onto the surface of the base.

Another method. We multiply the diameter of the sphere by 23;50 : 22 : 41 : 26 fifth to get the radius of the insphere. Then, we multiply one third of that height by the sum of the surfaces of that solid, the result is the volume of the solid. If the side of the base's triangle is known and the diameter of the sphere

بجهولا نقسم مقدار الضلع على وتر خمس الدائرة وهو $\frac{1}{2}$ ل
 كما مر سادس على ان نصف قطرها واحد فخرج نضرب بربعه
 في النخبة دائما فاحاصل مربع قطر الكرة الخارج التي يحيط بالمجسم
نوع آخر نقسم الضلع على $\frac{1}{2}$ ل لالب لرند $\frac{1}{2}$ خامسة يخرج القطر
 وامت الخامس فهو دواثنتي عشرة قاعدت محضات متساويت
 الاضلاع والزوايا وقع في الكرة والعمل فيه ان نأخذ نصف سدس
 مربع القطر ويحصل جذره ثم نضرب ذلك اعني نصف السدس
 المذكور في $\frac{1}{2}$ ل دائما ونأخذ جذر الحاصل وننقص منه الجذر السابق
 فمابغ فهو ضلع محض القاعد نوع آخر نضرب القطر في $\frac{1}{2}$ ل كما مر
 كما مر خامسة يحصل ضلع محض القاعد يحصل منه مساحه سطح
 القاعد كما سبق ونضرب في اثني عشر ليحصل مساحه جميع سطح
 في اثنتي عشرة قاعد ثم يحصل نصف قطر الكرة الداخلة كما سبق
 في دى عشرين قاعد بعينه اعني تنقص ثلث مربع ضلع المثلث في
 دى عشرين قاعد عن ربع مربع قطر الكرة المحيط ونأخذ جذر الباقي او
 نضرب القطر في $\frac{1}{2}$ ل كما مر خامسة فاحصل فهو العمود الخارج
 عن مركز المجسم الى مركز القاعد نضرب ثلثه في مساحه سطح المجسم يحصل
 جسم وهو المط وان كان ضلعه معلوما وقطر الكرة المحيط بجهولا نضرب الضلع
 ونزيد على ذلك المربع ربعه ونأخذ جذر المجموع وننقص عنه نصف الضلع فمابغ
 نزيد على الضلع المعلوم ونضرب مربع ما بلغ في الثلثة دائما فاحاصل هو

is unknown, we divide the length of the side by the chord of one fifth of the circle which is $1; 10 : 32 : 03 : 13 : 54 : 22$ sixth, assuming the radius is one. We then multiply the square of the result always by five. The result is the square of the diameter of the circumsphere that surrounds the solid.

Another method. We divide the side by $0; 31 : 32 : 37 : 54 : 13$ fifth to get the diameter.

The fifth is the solid with twelve pentagon bases with congruent sides and angles [regular dodecahedron], fitting in a sphere. The way to calculate it is to take half one sixth of the square of the diameter and obtain its square root. Then, we multiply the mentioned half sixth always by five, and take the square root of the result. Then, we subtract the earlier root from it, what results is the side of the pentagon.

Another method. We multiply the diameter by $0; 21 : 24 : 33 : 34 : 17$ fifth to get the side of the pentagon base. We obtain the area of the base from it as done earlier, then multiply it by twelve to get the surface area of the solid with twelve bases. Then, we find the radius of the incircle like we have done earlier in the solid with twenty bases [icosahedron], i.e., we subtract one third of the square of the side of the triangle in the icosahedron from one fourth of the square of the diameter of the circumsphere. Then, we take the square root of the remainder, or, we multiply the diameter by $23; 50 : 22 : 41 : 26$ fifth. The result is the height coming out of the center of the solid onto the center of the base. We multiply one third of it by the surface area of the solid to get its volume, which is the wanted value. If the side is known and the diameter of the circumsphere is unknown, we square the side, and add one fourth of that square to the square itself. Then, we take the root of the sum and subtract half of the side from it. We add the remainder to the known side, and multiply the square of the sum always by three, the result is

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مربع قطر الكرة التي يحيط بالمجسم **طريق** نقسم الضلع على ما سلكناه
 لدراسة يحصل قطر الكرة المحيطة ولما كان كل واحد من عدد قواعد
 هذا المجسم وعدد زوايا ذي عشرين قاعدة اثني عشر وعدد زواياها
 وقواعد عشرين فيمكن ان نعمل احدهما في الاخر بحيث تماس زوايا المجسم
 الداخل وكر اضلاع الخارج فيكون الكرة المحيطة بالمجسم الداخل المماسه لزواياه
 في الكرة الداخلة للمجسم الخارج المماسه لمراكز قواعد وكذا الحكم في
 المكعب وذي ثمانية قواعد وقد عرفت استخراج قطر الكرة
 الداخلة مما سبق وهي الكرة الخارجة للمجسم الداخل فاستخرج به
 ضلع مجسم الداخل ومساحته كما ذكرنا **واما السادس** فهو ذو اربع
 عشره قاعدة ثمانية منها مثلثات متساويات الاضلاع وستة
 الباقية مربعات اضلاعها اضلاع المثلثات وكل واحد منها مساو
 لنصف قطر الكرة المحيطة به والعمل فيه ان يضرب جذر نصف مربع
 القطر في ربع القطر اعني قاعدة المربعه ونحفظ الحاصل ثم نأخذ
 ثلث مربع القطر وكذا سدسه ويحصل جذر كل واحد منهما فالاول
 اربعة امثال العمود الخارج عن مركز مثلث القاعدة الاله نصف ضلعه
 والثاني العمود الخارج عن مركز المجسم الى مركز المثلث فنضرب نصف
 قطر الكرة وهو ضلع المثلث في احدهما ثم الحاصل في الاخر فالحاصل
 نزيد على المحفوظ فما بلغ فهو مساحه المجسم **طريق** احضرب القطر
 في ١٤ لولا منه ح خامسه والحاصل في مربع القطر فالحاصل هو

the square of the diameter of the circumsphere.

Another method. We divide the side by 0; 21 : 24 : 53 : 34 : 17 fifth to get the diameter of the circumsphere.

Since both the number of bases in this solid and the number of vertices in the solid with twenty bases are equal to twelve, and the number of vertices of this as well as its bases are twenty, it is possible to construct one of them inside the other, such that the vertices of the inner solid intersect with the centers of the sides of the outer one. So, the circumsphere of the inner solid passing through its vertices is the insphere of the outer solid, which is tangent to its sides at the midpoints. This also applies to the cube and the regular octahedron. You now know how to obtain the diameter of the insphere from what we had done before, which is the circumsphere of the inner solid. From that, you can obtain the side of the inner solid and its volume as we mentioned.

The sixth solid is the one with fourteen bases, eight of which are equilateral triangles, and the remaining six are squares with sides congruent to sides of the triangles, and each one of them is equal to the radius of the circumsphere [regular tetradecahedron]. The way to calculate it is to multiply the square root of half the square of the diameter by one fourth of the square of the diameter, i.e., its square base, and we save the result. We then take one third of the square of the diameter, as well as its sixth, and obtain the square root of each. The first value would be four times the height coming out of the center of the triangle base onto the middle of its side, and the second value is the height coming out of the center of the solid onto the center of the triangle. We multiply the radius of the sphere which is the side of the triangle by one of them, then the result by the other. We then add the result to the saved value to get the volume of the solid.

Another way. We multiply the diameter by 0; 10 : 26 : 23 : 45 : 58 fifth, and multiply the result by the square of the diameter. We call the result the saved value.

Then, we multiply the diameter by 0;16 : 19 : 47 : 45 : 13 fifth and multiply the square of the diameter by 0 : 25 : 58 : 50 : 44 : 37 fifth. Then, we multiply the first product by the second one and add the product to the saved value to get the volume.

The seventh solid is the one with thirty-two bases, twenty of which are equilateral triangles, and the other twelve are pentagons with sides congruent to the sides of the triangles [regular icosidodecahedron], which are equal to the sides of the decagon that can be constructed in a great circle of the sphere. The way to calculate it is to divide the square of the sphere's diameter by sixteen, and take the square root of the result. Then, we multiply the result of the division by five, take its square root and subtract it from the previous root. The remainder is the side of the base of the solid. We obtain from it the areas of both of its bases, i.e., the pentagon and the triangle as mentioned earlier in the area of surfaces. We multiply the area of the pentagon by twelve to get the sum of the areas of all pentagon bases, and multiply the area of the triangle by twenty to get the sum of the areas of all triangle bases. Then, we subtract one third of the square of the side from one fourth of the square of the diameter. We take the square root of the remainder and multiply its third by the sum of the areas of all triangular surfaces of the solid and save the result.

Then, we divide the side by 1;10 : 32 : 3 : 13 : 44 fifth. We subtract the square of the result from one fourth the square of the diameter, take the square root of the remainder and multiply its third by the sum of the areas of all pentagonal surfaces. We add the result to the saved value to get the volume of the solid.

Another method. We multiply the diameter of the sphere by 0;18 : 32 : 27 : 40 : 15 fifth to get the side. We obtain from it the areas of the pentagon and the triangle, and sum the areas of the pentagons first and the triangles second. Then, we multiply the diameter by 0;8 : 30 : 23 : 21 : 50 fifth, multiply the result by the sum of the areas of the pentagons, and save the result. Then, we multiply the diameter by

0; 9 : 20 : 30 : 12 : 18 fifth, and multiply the result by the sum of the areas of the triangles. [We add the product to the saved value to get the volume.]⁷⁰ [If the side is known]⁷¹ and the diameter is unknown, we do not take a quarter of the square of the side, but do take its square root. Then, we add the mentioned quarter to the square of the side and take the square root of the sum. Then, we subtract the previous root from that sum, and add the remainder to the side. Two times the result is the diameter of the circumsphere.

Another method. We divide the side by 0; 18 : 32 : 27 : 40 : 15 fifth to get the diameter.

The volumes of these solids with equal base sides were not mentioned by the masters of this art in the books of measurement, so I obtained it from the basic principles, and I put the numbers used in obtaining the areas in a table with the names of these numbers. The table is here:

⁷⁰ This is missing in the text but it is found in manuscripts [3, 4, 5], for example.

⁷¹ This is missing in the text but it is found in manuscripts [3, 4, 5], for example.

	Degrees	Minutes	Seconds	Thirds	Fourths	Fifths
Side of a regular tetrahedron assuming the diameter of the sphere is one and its height assuming the side is one	0	48	59	23	15	41
	Zero	Forty eight	Fifty nine	Twenty three	Fifteen	Forty one
Height of the triangle of a regular tetrahedron and the side of a regular octahedron assuming the diameter is one	0	42	25	35	3	53
	Zero	Forty two	Twenty five	Thirty five	Three	Fifty three
Diameter of the sphere of an octahedron assuming the side is one	1	24	51	10	7	46
	One	Twenty four	Fifty one	Ten	Seven	Forty six
Side of the cube assuming the diameter of the sphere is one	0	34	38	27	39	29
	Zero	Thirty four	Thirty eight	Twenty seven	Thirty nine	Twenty nine
Ratio of the side of the pentagon to the side of the hexagon	1	10	32	3	13	44
	One	Ten	Thirty two	Three	Thirteen	Forty four
Side of an icosahedron assuming the diameter is one	0	31	22	37	54	13
	Zero	Thirty one	Twenty two	Thirty seven	Fifty four	Thirteen
Height coming out of the center of an icosahedron or a dodecahedron on the base assuming the diameter is one	0	23	50	22	41	26
	Zero	Twenty three	Fifty	Twenty two	Forty one	Twenty six
Side of the dodecahedron assuming the diameter is one	0	21	24	33	34	17
	Zero	Twenty one	Twenty four	Thirty three	Thirty four	Seventeen
Half of the height coming out of the center of a tetrahedron on a square surface assuming the sphere's diameter is one	0	10	36	23	45	58
	Zero	Ten	Thirty six	Twenty three	Forty five	Fifty eight
Two-thirds times the height coming out of the center of a tetradecahedron on a triangular surface assuming the sphere's diameter is one	0	16	19	47	45	13
	Zero	Sixteen	Nineteen	Forty seven	Forty five	Thirteen
Ratio of the area of a triangle to the square of its side	0	25	58	50	44	37
	Zero	Twenty five	Fifty eight	Fifty	Forty four	Thirty seven
Side of an icosidodecahedron	0	18	32	27	40	15
	Zero	Eighteen	Thirty two	Twenty seven	Forty	Fifteen
A third of the height coming out of the center of an icosidodecahedron on its pentagon surface assuming the sphere's diameter is one	0	8	30	23	21	50
	Zero	Eight	Thirty	Twenty three	Twenty one	Fifty
A third of the height coming out of the center of an icosidodecahedron on its triangular surface assuming the sphere's diameter is one	0	9	20	30	12	18
	Zero	Nine	Twenty	Thirty	Twelve	Eighteen

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الفصل الثامن في مساحة سائر الاجسام اما المركبة مما ذكرنا
 مثلا اسطوانة تزيد عليه مخروط او نقص منه وامثال ذلك فيمسح كل
 واحد منها ثم نجعلها اونا هذا التفاضل على ما يقتضيه واما ما عدا ذلك
 فان امكن وضعه في الماء او حوض يمكن مساحت تجويفه نصفه
 ونصب عليه الماء الى ان جاوز الماء عن راسه ونعلم على الفضل
 المشتهر بين سطح الماء والانهاء او الحوض علامه ثم يخرج الجسم
 الماء ويمسح الهواء الواقع في الموضع الذي انخفض عنه الماء فهو الخط
الباب الثامن في معرفة مساحة بعض الاجسام

عن وزنه وبالعكس هي موقوفة على معرفة هذه المقدمه اذا كان جسمان
 متساويان في الحجم مختلفان في الوزن فان نسبة الوزن الاول
 الى وزن الثاني عند تساوي حجميهما كنسبة حجم الثاني الى حجم الاول عند
 تساوي وزنيهما مثلا يكون نسبة وزن الحديد الى وزن الخشب
 عند تساوي حجميهما كنسبة حجم الخشب الى حجم الحديد عند تساوي
 واكملي في معرفة هذه النسبة بين الاجسام المنطقه وغيره ما ان اخذ
 قمع يكون انبوتها منحنه مائده الراس الى اسفل فملأها ماء
 صافيا ونضع كفه ميزان تحتها فاذا اسقطنا او اولجنا فيها شيئا
 من الغلات او الاجزاء او غير ذلك وينبغي ان يكون مصمتا لا
 مجوفا فخرج من الانبوت بقدر حجم ذلك الجسم ماء واذا اسقطنا
 منها جسما اخر يكون وزنه مساويا لجسم الاول فخرج منها مقدار

Eighth Section. On the volumes of solids in general. As for composite solids that are constructed from solids we have mentioned, for example a cylinder with a cone added to it or removed from it, etc., we calculate the volumes of each, then we add them or subtract them according to what needs to be done. As for other solids, if it is possible to put the solid in a container or a basin whose volume can be calculated, we put the solid in it and pour water on it until the solid is completely submerged in water. We put a mark on the waterline in the container or the basin, then we remove the solid from the water and calculate the volume of the space where water receded from to get what we want.

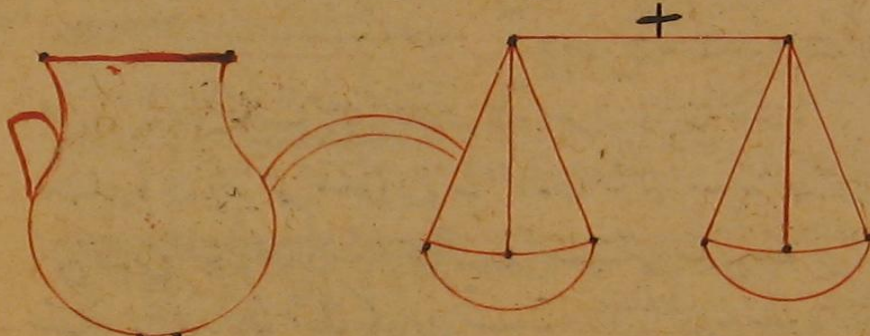
Eighth Chapter

On finding volumes of some solids from their weights and vice versa

This knowledge is dependent on knowing this introduction.

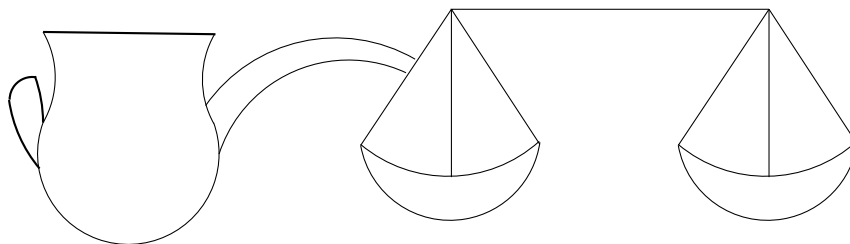
If there are two solid objects with equal volumes and different weights, then the ratio of the weight of the first one to the weight of the second when the volumes are equal is the same as the ratio of the volume of the second to the volume of the first when the weights are equal. For example, the ratio of the weight of iron to the weight of wood, when their volumes are equal, is equal to the ratio of the volume of wood to the volume of iron when they have equal weights. The trick in knowing this ratio between forgeable solids and unforgeable ones is to take a flask whose tube is bent downwards and fill it with clear water and put it on one side of a scale. If we drop or put in it some particles or gems, etc. that are full and not hollow, then some water will get out of the tube as much as the volume of that object. If we drop another object in the flask whose weight is equal to the weight of the first object, then another amount

او من الماء فيكون نسبة وزن الماء الاول الى وزن الماء الثاني
 كنسبة حجم الماء الاول بل حجم الجسم الاول الى حجم الماء الثاني بل
 حجم الجسم الثاني وكذا يكون النسبة بين وزن الجسم الثاني الى
 وزن الجسم الاول عند تساوي حجمهما فاذا استقطننا في القيمة
 ماية مثقال مثلاً من كل واحد من الاجسام التي ستوردنا في
 الجدول وبوزن ماء كل واحد يحصل لنا نسبة حجم بعضها مع
 عند تساوي الوزن



بل نسبة وزن بعضها مع بعض عند تساوي الحجم بالتكافؤ و
 لاستخراج نسبت المايعات ينبغي ان نأخذ ونعرف كم يسع ماء
 وهكذا كم يسع كل ما يبع لنعرف نسبة وزن الماء الى وزن كل واحد
 منها عند تساوي الحجم وقد عرفت نسبة وزن الماء الى وزن احد
 من الفلزات عند تساوي حجمها فنعرف نسبة وزن ذلك الفلز
 الى وزن كل واحد من المايعات عند تساوي الحجم ولو اردنا معرفة
 وزن مكعب ذراع من كل واحد منها نطلب بركة يكون جدرانها اما

of water will come out. The ratio between the weight of the first amount of water to the second amount equals the ratio of the volume of the first amount of water, i.e., the volume of the first object, to the volume of the second amount, i.e., the volume of the second object. This also equals the ratio of the weight of the second object to the weight of the first one when their volumes are equal. For example, if we drop a hundred Mithqāl⁷² in the flask of each of the materials we are going to mention in the table, and weigh the water [resulting] from each, then we get the ratio of the volumes of the materials if they had the same weight.



We get rather the ratio of the weight of one to another when the the volumes are equal in proportions. To extract the ratios of liquids, we need to take a container and find how much water it can hold, and similarly how much it can hold of every other liquid in order to know the ratio of the weight of the water to the weights of each of the liquids when the volumes are equal. You have already known the ratio of the weight of water to the weight of one of the metals when their volumes are equal. So, you can know the ratio of the weight of that metal to the weight of each of the liquids when the volumes are equal. If we want to know the weight of a cubed cubit of each of them, we find a tank with walls that are

⁷² The Mithqāl is the weight of one gold Dinar which is about 4.25 grams and is used to weigh precious metals and goods.

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مستویة او مستدیر قائمہ علی سطح الافق کل واحد من ابعادہا الثلثہ
 اکثر من ذراع وکلما كانت البرکۃ اعظم یکون العمل بها اصح ثم یلکما ما
 ونعلم الفصل المشترک بین سطح الماء وجدران البرکۃ ثم یمخرج منها
 بعضا من الماء بقدر ما ینخفض بہ سطح الماء من العلامة ذراعاً
 واحد و یوزن ما ینخرج منها ثم تقسم وزن الماء الذی اخضاه علی
 مساحة سطح الماء یحصل وزن مکعب ذراع من الماء ونسجج منه
 وزن مکعب کل جنس نرید علی نسبة وزنها عند تساوی الحجم وقد
 اورد الحکیم المحقق عماد الدین اکوام البغدادی تعذیرہ
 اللہ تعالیٰ بغفرانہ فی الرسالۃ البہائیتہ جدولین فی نسب الفلزات
 وایکواہر وبعض المایعات مستخرجین عن غیر ان الحکمتہ
 وہما غیر صحیحین فی کثیر من النسخ الّتی طالعہا لیسہو الناسخین
 ولم ینقض ذلك احد من شارحہ وقال الفاضل المحقق
 کمال الدین الحسن الفارسی فی الشرح ان لاسبیل لنا
 الی تصحیح اکد اول ونحن صححنا ما عن کتاب غیر ان حکمتہ
 فی ذکرنا کیف استخراجہا
 ایضاً لمن اراد
 استخراجہا
 واوردنا
 جدولاً
 واحد اما ما یتی ذی الفان

straight or curved, perpendicular to the horizon, each of its three dimensions is more than a cubit. The bigger the tank is, the more accurate the calculations performed with it are. Then, we fill the container with water and mark the level of the water on its walls. Then, we remove enough water so that the water level goes down by one cubit. We weigh the removed water then divide the weight by the surface area of the water to get the weight of a cubed cubit of water. We extract from it the ratio of the weight of a cube for each material we want over the weight of water when the volumes are equal. The wise commentator Imad al-Din al-Khawam al-Baghdadi, may God shower him with His forgiveness, presented in the Baha'i Treatise two tables on the ratios of minerals and gems and some liquids, which he extracted from the book *Mizan al-Hikma* [Balance of Wisdom]. The tables are incorrect in many copies due to scribe's oversight. This was not mentioned by any of the book's commentators. The righteous commentator Kamal al-Din al-Hasan al-Farsi said in the explanation: "There is no possible way for us to correct the tables." By the grace of God, we were able to correct them from *Mizan al-Hikma*, and we also mentioned how we extracted them for whomever wants to check them. We also present a table with the weights of the objects with equal volumes, considering the weight of the heaviest one, which is gold, to be a hundred, whether it is an ounce, a pound, or other weight.

[We also present another table in which] we consider the weight of gold to be two thousands and four hundred, as it is a mixed [value] of one hundred *Tasuj*s⁷³ and the weights of the waters of the objects considering the weight of each to be either a hundred or two thousand four hundred. We also convert it to the *jummal* system, because if a mistake is made during copying one, it is easy to correct it from another. Similarly, we present the weight of a cubed cubit using weight units, and in pounds as well. All of these values are averages. Here are the tables ⁷⁴:

⁷³ Originally, this was a weight value which became a division of the Islamic currency, the Dinar in copper.

⁷⁴ In the first table that follows, the rows in the manuscript that correspond to the gray cells do contain entries, however, they have been crossed-out and the words "blank is correct" have been written over them. These rows are empty in the manuscripts [3, 6], and "blank is correct" is written in [3]. This is understandable as al-Kashi is computing the water weight corresponding to a given volume of a solid object rather than a liquid. So, the entries for the liquids are empty. This table and the one following do not appear in manuscript [5].

اوزان المياه ماساوي حجم مائة مثاقيل اوغيا من كل جسم ومجنس طاسا يتجهما				اوزان الاجسام المتساوية الحجم على ان وزن الذهب مائة مثقال او اوقية او غيبي ومجنس طاسا يتجهما			
الوزن	القياس	القياس	القياس	الوزن	القياس	القياس	القياس
الذهب	١٥	١٣٩	و	٢	٢	٢	٢
الزيت	١٧٧	١	ز	١	١	١	١
الاسراب	٢١٢	٢	د	٢	٢	٢	٢
الفضة	٢٣٣	١	خ	٢	٢	٢	٢
الصفير	٢٧٢	٢	س	٢	٢	٢	٢
النحاس	٢٧٩	٢	ل	٢	٢	٢	٢
الشبه	٢٨٥	٢	م	٢	٢	٢	٢
الحديد	٣١٥	٢	ك	٢	٢	٢	٢
الرماس	٣٢١	٢	ح	٢	٢	٢	٢
الفاوق المحلى	٤٥٤	٢	و	٢	٢	٢	٢
المينا	٤١٥	٢	ل	٢	٢	٢	٢
الفاوق الاحمر	٤٢٤	٢	د	٢	٢	٢	٢
الفل	٤٧٥	٢	ك	٢	٢	٢	٢
الزهر	٨٧٢	٢	د	٢	٢	٢	٢
اللاجورد	٨٩٢	٢	د	٢	٢	٢	٢
اللولو	٩٢٤	٢	د	٢	٢	٢	٢
العقيق	٩٣٤	٢	د	٢	٢	٢	٢
السدر	٩٣٩	٢	د	٢	٢	٢	٢
البلور والحزع	٩٤٥	٢	د	٢	٢	٢	٢
الرجاج	٩٤٥	٢	د	٢	٢	٢	٢
الاسبنوس	١١٢	٢	د	٢	٢	٢	٢
المعاج	١٢٤	٢	د	٢	٢	٢	٢
الفل	١٧٣٨	٢	د	٢	٢	٢	٢
حسن البق	٢١٩٠	٢	د	٢	٢	٢	٢
حل الخمر	٢٢٢	٢	د	٢	٢	٢	٢
الحجر	٢٣٠	٢	د	٢	٢	٢	٢
الماء	٢٣٠	٢	د	٢	٢	٢	٢
السم	٢٣٠	٢	د	٢	٢	٢	٢
الزيت	٢٣٠	٢	د	٢	٢	٢	٢
عود الخلف	٢٣٠	٢	د	٢	٢	٢	٢

	Water weight of a volume of hundred Mithqāls or other units of each object and conversion to Tasujs				
	Mithqāls or Ounces	Daniqs	Tasujs	Conversion to Tasujs	Elevate in Jummāl Notation
Gold	Five	One	Two	126	Two Six
Mercury	Seven	Two	One	177	Two Six
Lead Sulphide	Eight	Five	Zero	212	Three Fifty-seven
Silver	Nine		One	233	Three Thirty-two
Topaz	Eleven	Two	Zero	272	Four Thirty-Two
Copper	Eleven	Four	Zero	276	Four Thirty-Six
Alum	Eleven	Ten	Zero	280	Ten Forty
Iron	Twelve	Five	Two	310	Five Ten
Lead	Thirteen	Four	Zero	328	Five Twenty-eight
Black Sapphire	Twenty-five	One	Two	606	Ten Ten
Enamel	Twenty-five	Two	Two	610	Ten Ten
Red Sapphire	Twenty-six	Zero	Zero	624	Ten Twenty-four
Carnelian	Twenty-six	Five	Two	670	Eleven Ten
Emerald	Thirty-six	Two	Zero	872	Fourteen Thirty-two
Azurite	Thirty-seven	One	Zero	892	Fourteen Fifty-two
Pearl	Thirty-eight	Three	Zero	924	Fifteen Twenty-five
Agate	Thirty-nine	Zero	Zero	936	Fifteen Thirty-six
Corals	Thirty-nine	Zero	Three	939	Fifteen Thirty-nine
Crystalline Crystals	Forty	Zero	Zero	960	Sixteen Zero
Glass	Forty	One	Zero	964	Sixteen Four
Ebony	Forty-six	Five	Zero	1124	Forty-eight Forty-four
Ivory	Sixty-one	Zero	Zero	1464	Twenty-four Twenty-four
Honey					
Cow Milk					
Wine Vinegar					
Wine					
Water					
Wax	Hundred	One	Zero	2524	Forty-two Four
Oil					
Willow Wood	Two Hundred Forty-eight	Zero	Two	5955	One Thirty-nine Fifteen

	Weight of objects with equal volumes considering the weight of gold to be one hundred Mithqāls or ounces or something else and conversion to Tasuj					It's Elevate in Jummal Notation		
	Mithqāls or Ounces	Danīqs	Tasuj	Its Fractions	Conversion to Tasuj	2 nd Elevate	1 st Elevate	Degrees
						Elevate	Degrees	Minutes
						Degrees	Minutes	Seconds
Gold	Hundred	Zero	Zero	Zero	2400	Forty	Zero	Zero
Mercury	Seventy-one	One	One	Twenty-eight	1708	Twenty-eight	Twenty-eight	Twenty-eight
Lead Sulphide	Fifty-nine	Two	Two	Twenty-five	1426	Twenty-three	Forty-six	Twenty-five
Silver	Fifty-four	Zero	Three	Fifty-one	1299	Twenty-one	Thirty-seven	Fifty-one
Topaz	Forty-six	One	Three	Forty-six	1111	Eighteen	Thirty-one	Forty-six
Copper	Forty-five	Two	Three	Forty-two	1091	Eighteen	Eleven	Forty-two
Alum	Forty-five	Zero	Zero	Zero	1080	Eighteen	Zero	Zero
Iron	Forty	Three	Three	Twenty-nine	0975	Sixteen	Fifteen	Twenty-nine
Lead	Thirty-eight	Two	One	Fifty-seven	921	Fifteen	Thirty-one	Fifty-seven
Black Sapphire	Four	Ten	Three	One	499	Eight	Nineteen	One
Enamel	Twenty	Three	Three	Forty-five	495	Eight	Fifteen	Forty-one
Red Sapphire	Twenty	One	Zero	Thirty-seven	484	Twenty-four	Four	Thirty-seven
Carnelian	Thirty-eight	Four	Three	Twenty-one	451	Seven	Thirty-one	Twenty-one
Emerald	Fourteen	Two	Two	Forty-seven	346	Five	Forty-six	Forty-seven
Azurite	Fourteen	Zero	Three	One	939	Five	Thirty-nine	One
Pearl	Thirty-three	Three	Three	Thirty-six	327	Five	Twenty-seven	Sixteen
Agate	Thirty-three	Two	Three	Six	323	Five	Twenty-three	Six
Corals	Thirty-three	Two	Two	One	322	Five	Twenty-two	One
Crystalline Crystals	Thirty-three	Zero	Three	Zero	315	Five	Fifteent	Zero
Glass	Thirty-three	Zero	One	Forty-two	313	Five	Thirty-three	Forty-two
Ebony	Eleven	One	One	Two	269	Four	Twenty-nine	Two
Ivory	Eight	Three	Two	Thirty-three	206	Three	Twenty-six	Thirty-three
Honey	Seven	One	Two	Thirty	174	Two	Fifty-four	Thirty
Cow Milk	Five	Five	Zero	Thirty-five	140	Two	Twenty	Thirty-five
Wine Vinegar	Five	Two	One	Twenty-two	129	Two	Nine	Twenty-two
Wine	Five	Two	Zero	Fifty	128	Two	Eight	Fifty
Water	Five	One	Two	Zero	126	Two	Six	Zero
Wax	Four	Five	Three	Forty-nine	119	One	Fifty-nine	Forty-one
Oil	Four	Five	Zero	One	116	One	Fifty-six	One
Willow Wood	Two	Zero	Two	Forty-five	50	Zero	Fifty	Forty-five

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عشرون
مواضع
صغيرة

وزن مكعب ذراع اليد بالمشاقيل و دقايقها									
ح	م	ث	د	ق	د	ق	د	ق	د
عود الخلف	١	٢	٣	٤	٥	٦	٧	٨	٩
الزيت	٩	٢	٣	٤	٥	٦	٧	٨	٩
الشمع	٢	٣	٤	٥	٦	٧	٨	٩	١٠
الماء	٨	٩	١٠	١١	١٢	١٣	١٤	١٥	١٦
الحجر	١٨	١٩	٢٠	٢١	٢٢	٢٣	٢٤	٢٥	٢٦
حل الجند	٥	٦	٧	٨	٩	١٠	١١	١٢	١٣
طست البقر	٤	٥	٦	٧	٨	٩	١٠	١١	١٢
العسل	٤	٥	٦	٧	٨	٩	١٠	١١	١٢
الرماض	١	٢	٣	٤	٥	٦	٧	٨	٩
الحديد	٣	٤	٥	٦	٧	٨	٩	١٠	١١
الذهب	١	٢	٣	٤	٥	٦	٧	٨	٩
النحاس	٧	٨	٩	١٠	١١	١٢	١٣	١٤	١٥
الفضة	٣	٤	٥	٦	٧	٨	٩	١٠	١١
الاسراب	١	٢	٣	٤	٥	٦	٧	٨	٩

وزن مكعب الذراع بالبرطل البغدادي و دقايقها									
ح	م	ث	د	ق	د	ق	د	ق	د
عود الخلف	١	٢	٣	٤	٥	٦	٧	٨	٩
الزيت	٢	٣	٤	٥	٦	٧	٨	٩	١٠
الشمع	٢	٣	٤	٥	٦	٧	٨	٩	١٠
الماء	٧	٨	٩	١٠	١١	١٢	١٣	١٤	١٥
الحجر	١٤	١٥	١٦	١٧	١٨	١٩	٢٠	٢١	٢٢
حل الجند	٤	٥	٦	٧	٨	٩	١٠	١١	١٢
طست البقر	١٤	١٥	١٦	١٧	١٨	١٩	٢٠	٢١	٢٢
العسل	٥	٦	٧	٨	٩	١٠	١١	١٢	١٣
الرماض	٩	١٠	١١	١٢	١٣	١٤	١٥	١٦	١٧
الحديد	٥	٦	٧	٨	٩	١٠	١١	١٢	١٣
الذهب	٢	٣	٤	٥	٦	٧	٨	٩	١٠
النحاس	٣	٤	٥	٦	٧	٨	٩	١٠	١١
الفضة	٢	٣	٤	٥	٦	٧	٨	٩	١٠
الاسراب	١	٢	٣	٤	٥	٦	٧	٨	٩

Object	Weight of an arm length cube in Mithqāls and its Fractions							Its Elevate in Jummal Notation				
	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units	Its Fractions	Mithqāls	1st Elevate	2nd Elevate	3rd Elevate	Mithqāls Fractions
Willow Wood	0	1	1	5	2	1	Forty-four	One	Twelve	Three	Zero	Forty-four
Oil		2	6	3	3	9	Ten	Fifty-Nine	Eighteen	Seven	Zero	Forty-four
Wax		2	7	2	0	2	Thirty-nine	Twenty-two	Thirty-three	Seven	Zero	Thirty-nine
Water		2	8	6	0	5	Forty	Forty-five	Fifty-six	Seven	Zero	Thirty-nine
Wine		2	9	2	4	8	Fifty-five	Twenty-eight	Seven	Eight	Zero	Forty
Wine vinegar		2	9	3	7	0	Zero	Thirty	Nine	Eight	Zero	Zero
Cow milk		3	1	9	1	6	Thirty	Fifty-six	Fifty-one	Eight	Zero	Zero
Honey	0	3	9	6	1	6	Thirty-five	Sixteen	Zero	Eleven	Zero	Thirty-five
Lead	2	0	9	3	0	8	Twenty-nine	Twenty-eight	Eight	Fifty-eight	Zero	Twenty-nine
Iron	2	2	1	4	0	3	Six	Three	Thirty	One	One	Six
Alum	2	4	5	1	9	1	Twenty-six	Thirty-one	Six	Eight	One	Twenty-six
Copper	2	4	7	8	4	7	Forty	Forty-seven	Fifty	Eight	One	Forty
Topaz	2	5	2	4	0	3	Twenty-three	Thirty-three	Six	Ten	One	Twenty-three
Lead sulphide	3	2	3	8	3	8	Five	Eighteen	Fifty-four	Twenty-eight	One	Five

Object	Weight of an arm length cube in Baghdad's pound						Its Elevate in Jummal Notation				
	Thousands	Hundreds	Tens	Units	Weight above the pound	Its Fractions	1st Elevate of the pound		Weights above the pound	Its Fractions	
Willow Wood	0	1	2	8	1	Forty-four	Two	Eight	One	Forty-four	
Oil	0	2	9	2	59	Ten	Four	Fifty-two	Fifty-nine	Ten	
Wax	0	3	0	2	22	Thirty-nine	Five	Two	Twenty-two	Thirty-nine	
Water	0	3	1	7	75	Forty	Five	Seventeen	Seventy-five	Forty	
Wine	0	3	2	4	88	Fifty-five	Five	Twenty-four	Eighty-eight	Fifty-five	
Wine vinegar	0	3	2	6	30	Zero	Five	Twenty-six	Thirty	Zero	
Cow milk	0	3	5	4	56	Thirty	Five	Fifty-four	Fifty-six	Thirty	
Honey	0	4	4	0	16	Thirty-five	Seven	Twenty	Fourteen	Thirty-five	
Lead	2	3	2	5	58	Twenty-nine	Thirty-eight	Forty-five	Fifty-eight	Twenty-nine	
Iron	2	4	6	0	03	Six	Forty-one	Zero	Three	Six	
Alum	2	7	2	4	31	Twenty-six	Forty-five	Twenty-four	Thirty-one	Twenty-six	
Copper	2	7	5	3	77	Forty	Forty-five	Fifty-three	Seventy-seven	Forty	
Topaz	2	8	0	4	43	Twenty-three	Forty-six	Forty-four	Forty-three	Twenty-three	
Lead sulphide	3	5	9	8	18	Five	Fifty-nine	Fifty-eight	Eighteen	Five	

ثم اذا كان مجسم معلوم الوزن وزيد مساحته نقسم وزنه على وزن مكعب ذراع
 منه يحصل المساحة واذا كانت مساحته معلومة وزيد الوزن فنضرب هذه
 وزن مكعب ذراع منه يحصل وزنه **الباب التاسع**
 في مساحة البنية والعمارات — ولم يذكر فيها اصحاب هذه الفن
 سوى الطاق والازج وذلك ايضا ليس على ما ينبغي فاوردتها على ما
 ينبغي مع سائره لان الاحتياج بمساحة العمارات اكثر من سائرها و
 جعلها مشتملة على ثلثة فصول **الفصل الاول** في مساحة الطاق
 والازج عرفها المتقدمون بانها نصف اسطوانة مستديرة محوفة ولانها
 مثله في العمارات القديمة والجديدة وما شابهها كان اكثره محددا لسط
 وقليل منه اقل من نصف الاسطوانة المستديرة المحوفة بكثرة قاعلم
 ان الطاق على ما ينبغي وهو ما يسمى بالطاق الحقيقي هو مستقيم مبني
 على قاعدتين هما في سطح واحد بين قطبين متوازيين كانه مؤلف
 من خمس قطعات اثنتان منها قطعان فلك واحد او ضلع واحد
 او دح واحد لا يكون قطر مقعرا اصغر من وسعة الطاق اعني البعد
 بين قاعدتي الطاق احدهما في اليمين والاخرى في اليسار مبنيان
 على القاعدتين وقطعتان احيان هما قطعان فلك او ضلع او دح في
 يكون قطر مقعرا اعظم من قطر متعر الفلك الاول وعللها مثل غلط
 القطعتين الاوليتين بعينه وهما مبنيان على فوق القطعتين الاولى
 متصلتان على خط هو محد الطاق ويكون محوري قطع اليمين في

Then, if we have an object whose weight is known and we want to know its volume, we divide its weight by the weight of a cubed cubit of it to get the volume. If its volume is known and we want to find the weight, we multiply the volume by the weight of its cubed cubit to get the weight.

Ninth Chapter

On the measurements of structures and buildings

The practitioners of this art have only mentioned vaults and arcs, and even that was not done properly. So, I cover them here in a proper way among others, because the need in buildings' measurements is more in demand⁷⁵. I made this chapter contain three sections.

First Section. On the areas of vaults and arcs. Previous scholars have defined them as a half of a hollow round cylinder, but we did not see alike [vaults or arcs] in old or new buildings. What we saw is mostly pointed middle [vaults and arcs] which are much less than a half of a hollow round cylinder. Know that a proper vault, which we call a real vault, is a structure that has a ceiling and is built on two bases on the same surface between two parallel lines. It consists of five pieces: two of which are parts of one spherical shell, or one annulus, or one timbrel with hollow diameter that is not smaller than the gap of the vault. I mean, [the gap is] the distance between the two bases of the vault, one to the right and the other to the left. In addition, [there are] two other parts of a spherical shell, or an annulus, or timbrel with hollow diameter larger than the hollow diameter of the first spherical shell, and their thickness is equal, exactly, to the thickness of the first two parts. They lay over the first two parts, connected by a line which is the definer of the vault. The axes of the two right parts are on

⁷⁵ In studying this chapter from an architectural point of view, Taheri observed that al-Kāshī's treatment of structures of buildings led to the first theoretical applications in architecture [41]. Moreover, Dold-Samplonius showed that al-Kāshī uses five methods for designing and drawing different types of pointed arches [19] and specific methods to approximate areas and volumes of domes [18].

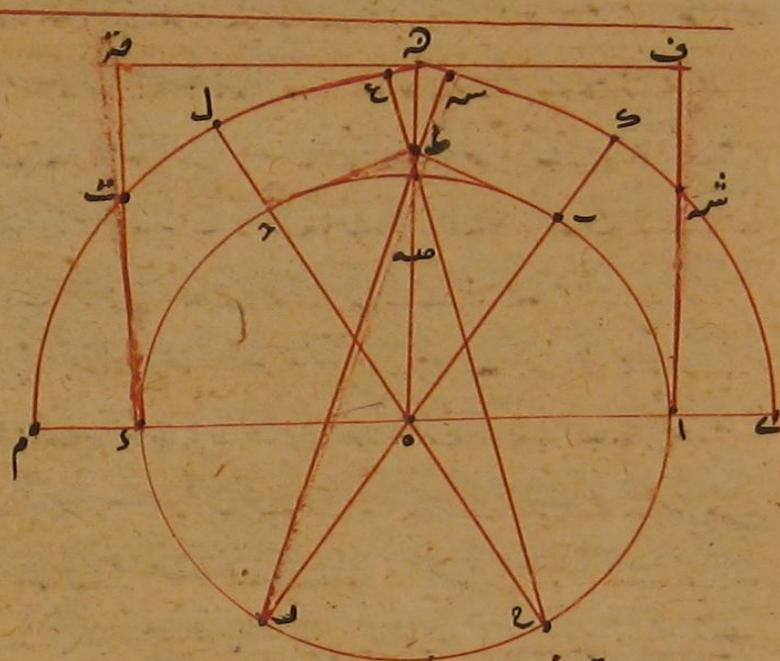
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سطح واحد وكذلك للأيام في سطح واحد آخر وقطعة واحد محيطها
 لوزيان متساويان متساويان متوازيان واربعه سطوح
 مستويات مجموعها هو مجسم محيط سطحان متساويان متوازيان
 وجهها و سطحان متساويان لعل محور واحد هما محدد ومقوده
 للبعد بين وجهيه عرض الطاق والفرق بين الطاق واللازج ان عرض
 الطاق لا يكون اكثر من وسعته ولللازج يكون اكثر منها وما يدعوه
 في الطاق عرض يدعوه في اللازج طول وطريق رسمه على ما راينا
 ختمه اوجه الاول ان ندير دائرة ا ب ج د على ان قطر ا ب يكون
 بقدر وسع الطاق ونقطه ه مركزها ونقسمها ستة اقسام متساوية
 على نقطه ا ب ج د ه ونصل اقطار ا ب ج د ه ونخرجها
 عن اطراف ا ب ج د ه على الاستقامة الى نقطه ك ل م
 بقدر سخن الطاق حسب ما نريد ثم ندير على مركزه قوسى
ك م ل وندير على نقطه ج بقدر ج ه قوس ج ه ط
 وعلى نقطه د بقدر د ه قوس د ه ط ونصل ج ط و د ط
 ونخرجها الى س بقدر سخن الطاق وندير على نقطه ج قوس
ك ه وعلى نقطه د قوس ك ه ونخرج عمود س ن على ط
س وعمود ن ج على ط ه فحصلت القطعات الخمس وهى
 قطعات ا ك ط ط ن ط ل ه جميعها وجه الطاق ولا جعلنا
س ن ه مستقيما لاستدراك القاعدة سنذكرها وصورتها هكذا

the same plane, and so are the axes of the two left ones, on another [plane]. Also, one part is surrounded by two similar congruent parallel right trapezoids, and four congruent plane surfaces with their combination being a solid surrounded by two equal parallel plane surfaces which are its faces, and two curved surfaces not on the same axis, which are its convex and concave parts. The distance between its faces is called the width of the vault. The difference between a vault and an arc is that the width of a vault is not bigger than its gap, but for the arc, it is bigger. What we call width in the vault is called length in the arc. According to what we have seen, there are five ways to draw vaults.

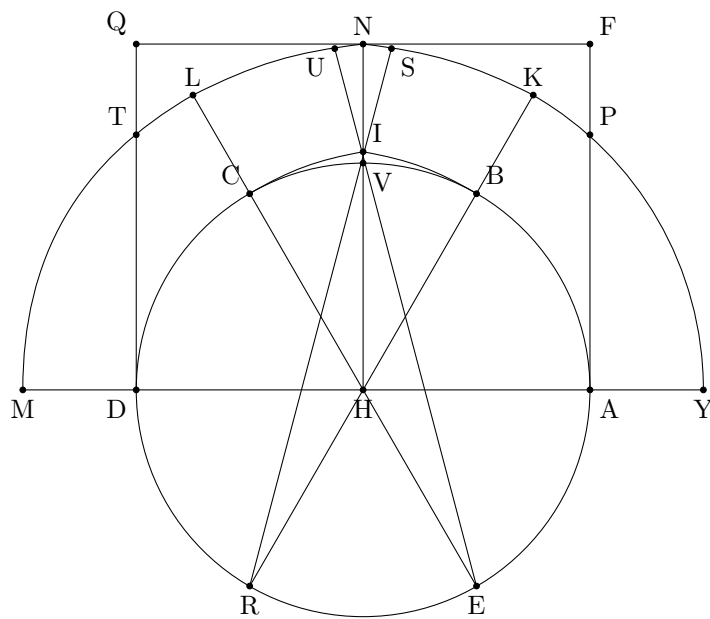
The first one is to draw a circle $ABCD$, such that its diameter is equal to the span of the vault, with a point H as its center. We divide it into six equal parts on the points A, B, C, D, R, E , and we connect the diameters AD, BR, CE . We extend those diameters perpendicularly from the sides AB, CD to the points Y, K, L , and M , as we wish to equal the width of the vault. Then, we draw around its center, the arcs of YK and ML . We also draw around the center E , with length EC , the arc CI and around the point R with a distance equal to the length of RB , the arc BI . Then, we connect EI and RI , and we extend them to points S, U equal to the width of the vault. Then, we draw the arc LU around the point E , and we draw the arc KS around the point R . We then draw a line SN perpendicular to IS , and another line, NU perpendicular to IU . Thus, we get the five parts, which are AK, KI, IN, IL , and LD , and they constitute a form of the vault. We made SN and UN straight and not curved for a reason that we will mention. This is its figure:

ويجوز ان
نقسم قس
ط ط ٦
ك س ٦
حول نقطتين
اخرين على
خطي ه ز
ه ح ا
داخل
خلف



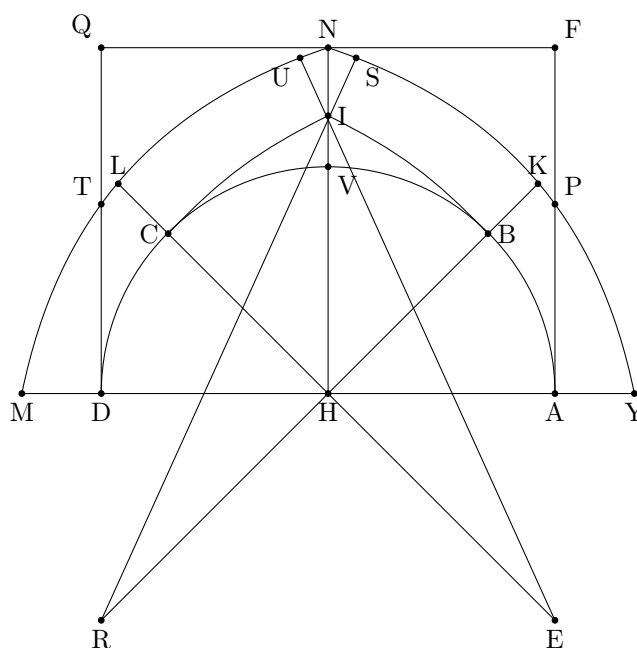
الدائرة التمامي واما خارجة والاحسن في سبقي ونسج سطح ا ب ط
و ك مجوف الطاق ونذاعه البناءون باسبيرة واذا اخرجنا من
نقطه ه في الجانين عمودي ن ف ن ق على ه ط ن مساوين
لاه ونصل اف و ك نقطتان متحدب الطاق على نقطتي
س ه ت فسطحي ش ف ه ه ق ت هما كتنا الطاق واشه
ه ك ت م ما وقع من الطاق في الحدار وخط ط ه ارتفاع محده
الاسفل وه ن ارتفاع محده الاعلى وهذا الوجه يليق حيث
كانت وسعة الطاق الى خمسة اذرع وقد شاهده في بعض
العمارات ان ط ط ٦ كانا خطين مستقيمين وكذا ان ن ك

اق
٦



It is permissible to draw the two domes of BI, CI , and KS, UL around a different pair of points on the lines HR and HE [respectively], either inside the lower semicircle or outside of it, which is better. We call $ABCDI$ the hollow of the vault, and the masons call it its depth. If we draw two lines from the point N on both sides, call them NF and NQ , perpendicular to HIN , equal to AD , and if we connect AF and DQ , that will intersect with the convex of the vault in two points P and T . So, the surfaces of PFN and NQT are the shoulders of the vault. APY and DTM are the parts of the vault that are in the wall. The line segment IH is the height of its lower definer, and HN is the height of its upper definer. This way of drawing is appropriate when the span of the vault is up to five cubits. In some buildings, we see that BI and IC are straight lines, and so are KN and NL .

The second way. We draw a semicircle $ABCD$ such that the line segment AD is the diameter, and it is the span of the vault. We extend it from both sides to two points, Y and M , such that the distance is equal to the width of the vault, as needed. The center of the circle is the point H . We divide the circle into four equal parts on the points A, B, V, C , and D , and we connect the radii BH and CH . Then, we extend them and single out HR and HE , so that their lengths are equal to the length of AV , the chord of the quarter of the circle. We also [extend CH and BH] into CL and BK of the same lengths as the width of the arc, i.e., DM . We draw two arcs around the center H , call them YK and ML . We draw an arc CI around point E , with a distance equal to the length of EC . We also draw an arc BI around a point R , with a distance equal to the length of RB . We connect EI and RI , and extend them to two points, U and S , with a distance equal to the width of the arc. We draw the arc LU around point E and draw the arc KS around point R . We draw two lines, SN and UN , perpendicular to IS and IU [respectively]. So, the combination of the parts AK, KI, IN, IL , and LD is the face of the vault. Then, we complete the surface of the parallelogram $AFQD$, and make SN and UN straight instead of round for a reason which will be understood. This way is appropriate when we need the span of the vault to be between five to ten or to fifteen cubits. This is the figure:

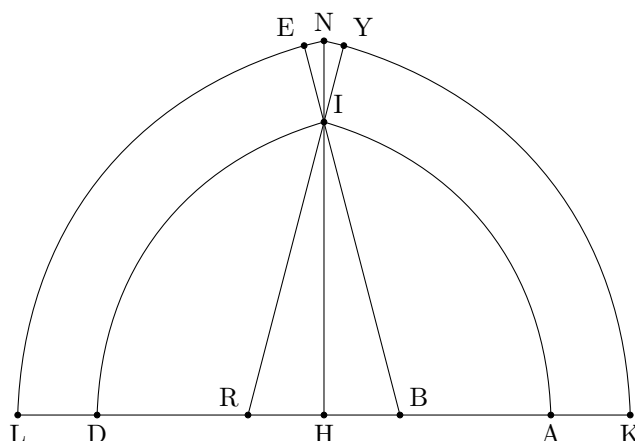


The diagram illustrates a geometric construction. A square $QNHD$ is shown with vertices Q (top-left), N (top-right), H (bottom-right), and D (bottom-left). A circle is drawn passing through points M , D , L , C , U , and N . A line segment EN is drawn, passing through points S , B , and A . Another line segment EN is drawn, passing through points V , I , and A . The intersection of the circle and the line EN is at point C .

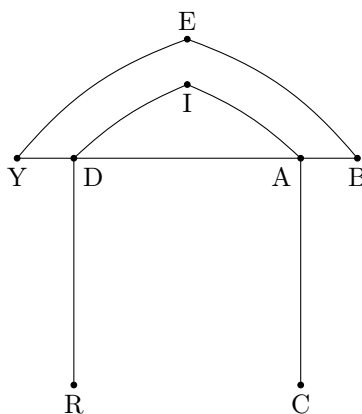
The fourth way. We trisect the span of the vault AD at points B and R . We draw an arc DI around point B with a distance equal to the length of BD , and we draw an arc AI around point R with a distance equal to the length of AR . We connect BI and RI , and we extend them to points E and Y [respectively] with a distance equal to the width of the gap. We extend AD similarly in both directions to points K and L . We draw an arc LE around point B with a distance equal to the length of BL , and we draw an arc YK around point R with a distance equal to the length of RK . From points E and Y , we draw two line segments, EN and YN [respectively], perpendicular to IE and IY [respectively]. The combination of

77 The Ba'a is the distance between the tips of fingers when the left and right arms are extended to the left and right respectively. It is about 6 feet.

the parts IK , IN , and IL together form the face of the arc like this⁷⁸:



The fifth way. We draw two line segments from A and D , the ends of the vault's span, call them AC and DR , such that the line segments are perpendicular to AD . We make each of these segments equal to the length of AD . We make points C and R into centers and draw two arcs around each of them, AI , DI , with a radius equal to the hypotenuse of the perpendiculars, i.e., equal to the length of AR . We do the same with the arcs and BE , YE ⁷⁹, after extending the segment AD by the same length in both directions. So, the figure $ABEYDI$ is the face of the vault like this.



Since we are now done with defining vaults and arcs, let us now start with calculating their area. We have extracted the ratios of some of its dimensions to its span, and their ratios to its width, and we put them in a table and explained how to use it. We will include how to extract these quantities. We also converted them to the Indian numerals and put them in a table as well. And here is the table.

⁷⁸ A note on the left margin in Persian: "In this figure, the position of [the point] I of intersection of the lines is a bit off, but that is not very important as it [the construction] is correct." Since this manuscript is written around 1450 CE, this note seems to be inserted by the scribe as there is not a similar note in subsequent manuscripts. Moreover, since the note is in Persian, it indicates that the scribe is most likely a Persian.

⁷⁹ Following the constructions of the previous arcs, the radius of the arcs here is the length of BR .

إذا ضربنا وسعة الطائفت في هذا فحصل معلوم الطائفت	إذا ضربنا سمي الطائفت في هذا فحصل المطلوب على معلوم الطائفت ونهت الجميع في تحسن الطائفت يحصل ما نريد	فحزب ووسعة الطائفت فحصل فحصل ارتفاع حذره الأسفل	بغير سمي الطائفت في هذا فحصل مطلوب على ارتفاع حذره الأسفل فحصل ارتفاع حذره الأعلى	فحزب جميع وسعة الطائفت في هذا فحصل مساحة سطح مجموع الزوايا الساكنة باستنباط
بالوجه الأول	بالوجه الثاني	بالوجه الثالث	بالوجه الرابع	بالوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس
الوجه الأول	الوجه الثاني	الوجه الثالث	الوجه الرابع	الوجه الخامس

وذلك المتأدير بالرقوم المنديّة

بالوجه الأول	بالوجه الثاني	بالوجه الثالث	بالوجه الرابع	بالوجه الخامس
بالوجه الأول	بالوجه الثاني	بالوجه الثالث	بالوجه الرابع	بالوجه الخامس
بالوجه الأول	بالوجه الثاني	بالوجه الثالث	بالوجه الرابع	بالوجه الخامس
بالوجه الأول	بالوجه الثاني	بالوجه الثالث	بالوجه الرابع	بالوجه الخامس
بالوجه الأول	بالوجه الثاني	بالوجه الثالث	بالوجه الرابع	بالوجه الخامس

	If we multiply the span of the vault by this result, we obtain the concave part of the face of the vault				If we multiply the thickness of the vault by this result, and add the result to the concave part of the face of the vault and multiply the sum by thickness of the vault, we get the area of its face			
	Degrees	Minutes	Seconds	Thirds	Degrees	Minutes	Seconds	Thirds
First Way	One	Thirty-seven	Twenty-six	Six	One	Thirty-five	Thirty-Seven	Twenty-eight
Second Way	One	Thirty-Nine	Twenty-Six	Six	One	Thirty-five	Thirty-seven	Twenty-eight
Third Way	One	Forty-two	Forty-four	Three	One	Thirty-four	Twenty-one	Forty-seven
Fourth Way	One	Forty-five	Twenty-six	Fifty-seven	One	Thirty-four	Thirty-four	Forty-three

Multiplying the span of the vault by this, we obtain the height of its lower definer				By multiplying the thickness of the vault by this result, we get the thickness of its definer. By adding it to the thickness of its lower definer, we get the height of its upper definer				Multiplying the square of the span of the vault by this, we get the area of the convex part called depth by builders			
Degrees	Minutes	Seconds	Thirds	Degrees	Minutes	Seconds	Thirds	Degrees	Minutes	Seconds	Thirds
Zero	Thirty-four	Seven	Thirty-eight	One	One	Fifty-eight	Four	Zero	Twenty-four	Twenty-eight	Forty-two
Zero	Thirty-five	Fifty-five	Sixteen	One	Five	Fifty-five	Twelve	Zero	Twenty-five	Nine	Thirty-three
Zero	Thirty-eight	Seventeen	Twenty	One	Six	Fifty-five	Thirty-eight	Zero	Twenty-five	Two	Thirty-four
Zero	Thirty-eight	Forty-three	Forty-seven	As One	in Five	the second Fifty-five	Exactly Twelve	Zero	Twenty-eight	Zero	Zero

Those Quantities in Indian Numerals

	Units Tenths Hundredths Thousandths	Units Tenths Hundredths Thousandths	Units Tenths Hundredths Thousandths	Units Tenths Hundredths Thousandths	Units Tenths Hundredths Thousandths
First Way	1 6 2 4	1 5 9 4	0 5 6 9	1 0 3 3	0 4 0 8
Second Way	1 6 5 1	1 5 9 6	0 5 9 8	1 0 9 9	0 4 1 9
Third Way	1 7 1 2	1 6 0 6	0 6 3 8	1 1 1 5	0 4 5 1
Fourth Way	1 7 5 7	1 5 7 6	0 6 4 5	1 0 9 9	0 4 7 8

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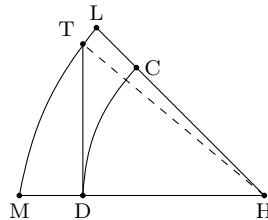
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فاذا حصل مساحة وجه الطاق من الجداول الثاني ضربها في عرض الطاق
 - حصل مساحة مجسمه ولما كان اكثره معمولاً بوجه الثاني فبالقريب انه اذا
 كان وسعة الطاق عشرين يكون مقع وجهه ثلثه وثلثين وارتفاع محده
 الاسفل اثنا عشره واذا كان ثلثه خمسة يكون محن محده خمسة ونصف ويكون
 نصف التفاضل بين المحرب والمقعر وجهه ثمانية فاذا اردنا نصف التفاضل
 على مقع وجهه وضربنا المجموع في ثلثه يحصل مساحة وجهه واذا ضربنا مربع
 وسعته في الثلثه دايماً وقسمنا الكااصل على اثنا عشره حصلت مساحة مقع وجهه
 المدعوباً نسبة **واتى** مساحة ما يدخل من الطاق في الجدار الذي بني عليه
 ومساحة كتفه ففرض نصف قطر مقعر القطعة الاولى منه وهو نصف
 وسعته في الوجهين الاولين ونصفها ونصف ثلثها في الوجه الثالث
 وثلثها في الوجه الرابع في نصف قطر محدها من خطا وهو مجموع ثلثه مع
 نصف قطر مقعها ونقوس الكااصل فيجب وناخذ تمامها فهو قوس من
 محده الطاق يدخل في الجدار من احد جانبيه تمامه المحيط لثلاثيه وستون ثم
 نضرب نسبة المحيط الى القطر في مجموع وسعة الطاق ونضع ثلثه في الوجهين
 الاولين ويزياده ثمن الوسعة في الثالث ويزياده ثلثها في الرابع فما حصل
 في القوس المذكور ونقسم الكااصل على ثلثها يه وستين فما خرج فهو مقدار
 القوس المذكور تمامه وسعة الطاق ممسوحاً بنضربه في نصف قطر محده
 القطعة الاولى فما حصل يحفظ ثم ناخذ جيب تلك القوس ونضربه في
 نصف القطر المذكور من خطا فما حصل نضربه في نصف قطر مقعر القطعة الاولى

So, when we get the area of the face of the vault from the second column, we multiply it by the width of the vault to get its volume. Since most of the table is done in the second way, we can approximate that if the gap is twenty, then the concave part would be thirty-three, and the height of its lower definer is twelve. Also, if its width is five, then the width of its definer is five and a half, and half the difference between the concave part and convex part is eight. So, if we add half the difference to the concave part and multiply the sum by its width, then we get the area of the vault's face. And if we multiply the square of its gap by five always, and divide the result by twelve, then we get the area of its concave part, also known as its depth.

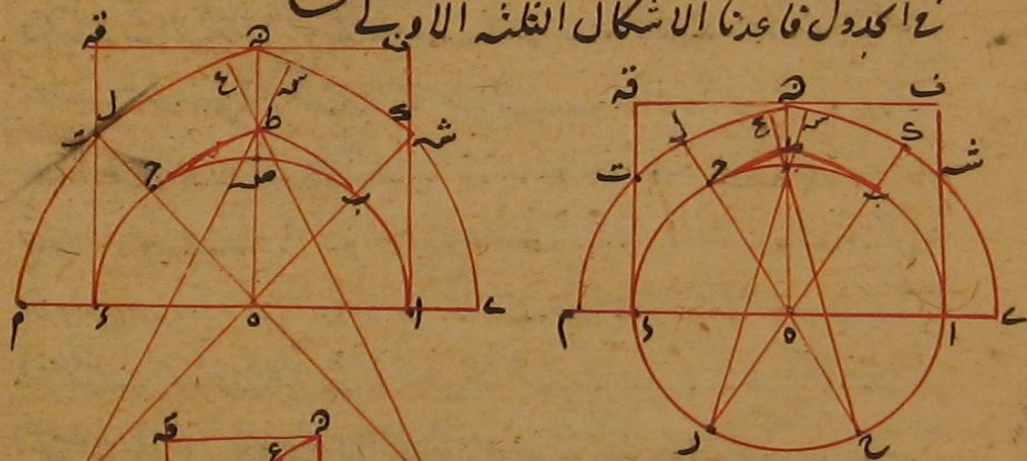
As for the area of the parts of the vault that go inside the wall on which it is built, and the area of its shoulder, we multiply ⁸⁰ the radius of the concave part of its first part, which is half of its span in the first two ways, a half and a half of an eighth of the span in the third way, two thirds of the span in the fourth way, by the radius of the convex part, reduced by a factor of sixty, which is the sum of the vault's width and the radius of its concave part. We take the arcsine of the result, then take its complement. What results is an arc from the convex part of the vault, which goes inside the wall from one of its sides, using a unit in which the circumference equals three hundred sixty. Then, we multiply the ratio of the circumference to the diameter by the sum of the vault's span and double its width in the first two ways, and by adding one eighth of the span for the third way and one third of it for the fourth. We multiply the result by the aforementioned arcsine, and divide the product by three hundred sixty. What results is the measurement of the mentioned arc, with the same unit used to measure the span of the vault. We multiply this result by the radius of the convex part of the first part and save the result. Then, we take the sine of that arc and multiply it by the aforementioned radius reduced by a factor of sixty. We multiply the result by the radius of the concave part of the first part,

⁸⁰ According to ([7], P. 179), and ([6], P. 365) we should divide instead of multiply. We can see that by looking at the portion of the vault that goes inside the wall. Following ([6], P. 365), according to the figure of the previously mentioned second way for example, the part that goes inside the wall on the left side is MTD . Here is a portion of that figure with the dashed line segment HT added to make a right triangle.



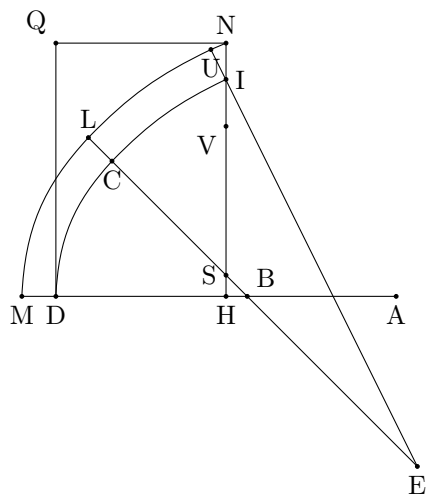
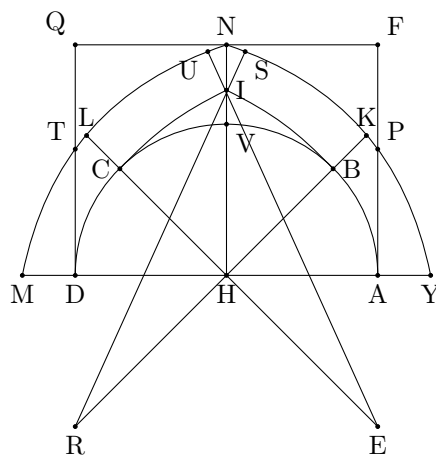
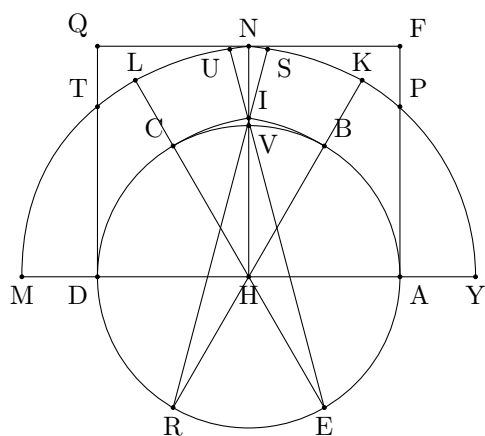
In the right triangle DTH , $\sin(T) = \frac{HD}{HT} = \frac{HD}{HM}$ which is the radius of the concave part of its first part HD over the radius of the convex part HM . Al-Kāshī said it is reduced by a factor of sixty, because his sine is 60 times our sine. So, he has to divide by 60.

فاحصلت انتقصه من المخفوظ فمابقي هو مجموع سطح القطعتين اللتين يدخل
 في الجدار انتقصه عن مساحة وجه الطاق فمابقي زيد على مساحة مجوفة و
 انتقص المجموع عن مضروب وسعة الطاق في ارتفاع محدود الاعلى فالباق
 فهو مساحة سطح كتيه ثم ضرب سطح كل واحد مما يدخل في الجدار من الطاق
 و سطح كتيه في عرض الطاق ليحصل مساحة جسمه والاولى في مساحة
 العمارات ان نسح الجدار الى ان نشاء الطاق اولاً ثم نسح الطاق ومجوفه
 ثم نضرب مجموع وسعة الطاق وضعف تحت في ارتفاع محدود الاعلى و انتقص
 من الحاصل مجموع مساحة وجه الطاق و سطح مجوفه فمابقي هو مساحة سطح
 كتيه مع ما وقع فوق قاعدته ليلا يحتاج الى مساحة ما يدخل في الجدار من
 الطاق واما اراد ما وعدناه في كتيه استخرج مقادير النسب الموضوعة
 في الجدول فاعدنا الاشكال الثلاثة الاولى



وفرضنا وسع الطاق اثنين وضربناه في نسبة ط في نسبة ط
 المحيط الى القطر حصل وبنوط ح اضناه

and subtract the result from the saved value. What remains is the sum of the areas of the two parts that go inside the wall. We subtract this value from the area of the face of the vault, and add the remainder to the area of its concave part. Then, we subtract the sum from the product of the vault's span and the height of its upper definer. The remainder is the area of its shoulders. Then, we multiply the surface area of each of the parts of the vault that go into the wall and the surface area of the vault's shoulders by the width of the vault to get the volume of the vault. It is better in calculating the volume of a building to measure the walls up to the beginning of the vault first. We then measure the vault and its concave part. Then, we multiply the sum of the vault's span and double its width by the height of its upper definer. From the result, we subtract the sum of the areas of the vault's face and the surface of its concave part. What remains is the surface area of the vault's shoulders with what lies above its base in order for us not to calculate the area of the vault that is inside of the wall. As for what we promised of including the way in which we extracted the quantities in the table, we repeat the first three figures.



We assume the span of the vault to be two, and we multiply it by the ratio of the circumference to the diameter and get 6 : 16 : 59 : 28. We then obtain

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في الوجه الاول		سدس الخلال	١- مطا	وهو شعاع واحد التقطعتين الاولين اعني احدى قوسي	٥٥٤	قنه	ل ٢ ٢ ٢
في الوجه الثاني		ثمن	٢- موركو	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الثالث		ثمن وثمن ثمن	٣- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الرابع		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الخامس		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه السادس		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه السابع		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الثامن		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه التاسع		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه العاشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الحادي عشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الثاني عشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الثالث عشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الرابع عشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الخامس عشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه السادس عشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه السابع عشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الثامن عشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه التاسع عشر		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه العشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الحادي والعشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الثاني والعشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الثالث والعشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الرابع والعشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الخامس والعشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه السادس والعشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه السابع والعشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الثامن والعشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه التاسع والعشرون		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥
في الوجه الثلاثين		٥٥٤	١- حركه	١- حركه ويكون زاوية	٥٥٤	قله	س ٥ ٥ ٥

ولما كان فضل محيط على محيط اخر على ان الفضل بين نصع قوسيهما واحد **ولو نط**
 ونسبة الثلثية وستين كنسبة فضل **ل** على **ط** اذا كان البعد بينهما واحد **ل** زاوية **ط** وهي

First Way	One-sixth of the result	1 : 2, 49, 59	It is the convave part of one of the first two figures, i.e., one of the arcs <i>ABCD</i> and it is an angle	8 : 5, 9	150	Sine	30 : 0, 0, 0
Second Way	Its eight	0 : 46, 7, 26		8 : 5, 9	135		45 : 25, 35, 4
Third Way	Its eighth and eighth of its eighth	0 : 53, 0, 52		8 : 60, 9	135		45 : 25, 35, 4

Multiply by the line	8 : 5	It is	1 : 0, 0, 0	We divide the result by the segment <i>EI</i> which is the radius of the second figure, and it is	2 : 0, 0, 0	We get from the division, the sine of the angle <i>EIH</i>	55 : 0, 0	Arcsine, which is the angle of the right kite	14 : 38, 39	Remains the angle <i>IEC</i>	55 : 31, 21
	8 : 5		1 : 24, 51, 10		8 : 5, 9		24 : 51, 10		24 : 28, 11		20 : 31, 49
	8 : 60		1 : 25, 26, 33		2 : 32, 21, 10		26 : 34, 59		26 : 28, 11		18 : 42, 5

It is the concave part of one of the last two figures with the circumference being three hundred sixty and the radius 57 : 17, 44, 49, according to [one of the following options]:

Radius	2 : 0, 0, 0	It becomes	34 : 36, 36	We add it to the arc <i>CD</i> to get arc <i>ICD</i>	1 : 37, 26, 6	Assuming half of the vault's span is one, we put it in the first table. If we multiply it by half of the vault's span, we get half of its concave part. And if we multiply by the span instead, we get all of its concave part	Then, we multiply the vault's thickness once, and add to the circumference 6 : 16 : 59 : 28. We take	Its sixth	1 : 2, 19, 55	It is the difference of <i>ML</i> and <i>CD</i> assuming <i>DM</i> , the vault's thickness is one
	2 : 24, 51, 11		51 : 54, 53		1 : 39, 2, 19				0 : 47, 30, 26	
	2 : 32, 21, 10		49 : 43, 11		1 : 42, 44, 3				0 : 47, 6, 26	

Since the difference between the two circumferences with radii differing by one is 6 : 16 : 59 : 28, and its ratio to three hundred sixty is equal to the ratio of the difference between *EL* and *IC* if the difference between the is one to the angle of *IEC*, which is:

First Way	15 : 31, 21	The difference of UL and CI is	0 : 17, 38, 5	Since the angle of the right kite is		The side UN is		We added it to the arcs ML and LU and get the difference of MLN and ICD	1 : 35, 37, 28
Second Way	20 : 31, 49		0 : 21, 29, 57						1 : 35, 52, 42
Third Way	18 : 42, 5		0 : 19, 35, 3						1 : 36, 21, 47

⁸¹ If we multiply the thickness of the vault by it, we get the difference of half its convex part and half its concave part. Actually, we get half the difference of all of its convex part and all of its concave part. We put it in a second table.

Then, we multiply EI by the sine of the angle IEC which is

2 : 0, 0, 0	We divide the result by the sine of angle EHN or ESN which is	30 : 0, 0, 0	The result of the division is segment	HI	1 : 8, 15, 16
2 : 24, 51, 10		42 : 35, 25, 4		HI	1 : 11, 50, 32
2 : 32, 21, 10		42 : 35, 25, 4		SI	1 : 9, 5, 1

It is the height of the lower definer in the first two ways. In the third, it is the difference of length of the segment HS to SE , 1 : 36, 35, 1. We halve it to get

0 : 34, 7, 38	We put it in the third table, then divide one over the sine of the complement of the angle of the right kite. It is	58 : 5, 40	The result of the division is the thickness of the vault's definer	1 : 1, 58, 4	It is put in the fourth table
0 : 35, 55, 16		54 : 36, 39		1 : 5, 54, 12	
0 : 38, 17, 30		53 : 47, 23		1 : 6, 55, 38	

Half of the span is one.

Then, we multiply by the radius of the concave part of the first figure by the arc CD . We get the double area of the sector

3 : 5, 4	1 : 2, 49, 55	Then, we put the radius of the second figure by the arc IC , which is	34 : 36, 36	The result is half the area of sector IEC	1 : 9, 12, 23	We multiply the height out of the point E of triangle EIC onto segment IN which is out of the triangle. It is	30 : 0, 0, 0	By the base of the triangle, which is	1 : 8, 15, 16
3 : 5, 4	0, 47, 7, 46		51 : 54, 53		2 : 5, 19, 59		1 : 0, 0, 0		1 : 11, 50, 32
3 : 5, 4	0 : 49, 38, 39		49 : 43, 11		2 : 6, 14, 56		1 : 6, 30, 0		1 : 9, 5, 1

Results of the double of the area of the triangle.

34 : 7, 38	We subtract it from the double of the area of sector IEC . Remains double the area	9 : 5, 3	35 : 5, 3	We add it to half the sector	3 : 5, 4	Results of the area of the vault's concave	1 : 37, 54, 59	If we make the square of the vault's opposite side one, that area will be this	24 : 28, 42
1 : 11, 50, 32		9 : 5, 3	13 : 29, 24		3 : 5, 4		1 : 30, 16, 53		25 : 9, 13
1 : 17, 43, 9		9, 60, 3	48, 31, 37		3 : 2, 4		1 : 47, 26, 4		27 : 2, 34

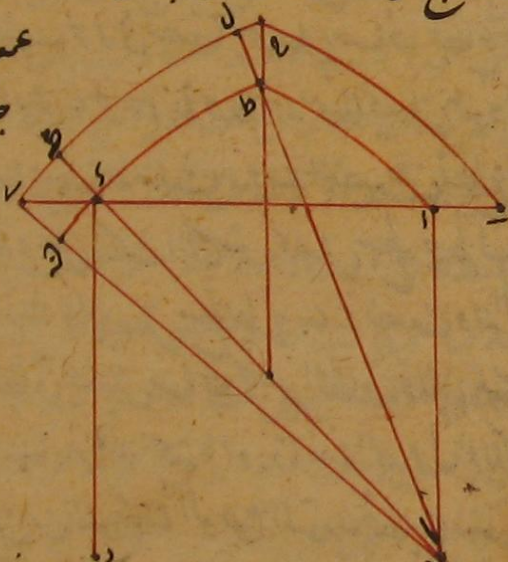
⁸¹ The above table is incomplete as some entries are empty. We complete it here as follows using manuscripts [4, 5].

First Way	15 : 31, 21	The difference of UL and CI is	0 : 17, 38, 5	Since the angle of the right kite is	14 : 28, 39	The side UN is	5 : 15, 29, 30	We added it to the arcs ML and LU and get the difference of MLN and ICD	1 : 35, 37, 28
Second Way	20 : 31, 49		0 : 21, 29, 57		24 : 28, 11		0 : 27, 28, 19		1 : 35, 52, 42
Third Way	18 : 42, 5		0 : 19, 35, 3		26 : 17, 55		0 : 29, 19, 39		1 : 36, 21, 47

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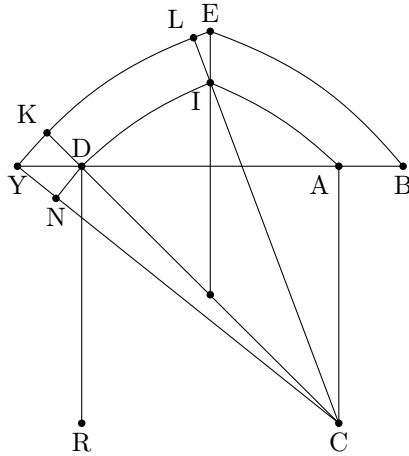
٩٣

وهو العدد الموضوع في الجدول الخامس فاذا عرفت استخرج تلك النسب
في الوجه الثالث فلا يخفى عليك الوجه الرابع لسهولة او نصف قطر قوس
مقوره بقدر ثلثي وسعته ونصف مقوره بقدر قوس يكون جيب تمامها
ثلث القطر واما مساحة الطاق بالوجه الخامس فيكتفي فيها ان نضرب
مربع وسعته في
ثالث الاشارة ليحصل مساحة سطح مجموعته نظر بان عرض الطاق ونقص
الحاصل مع ما تحته من التجويف عن مساحة الجدار لان وقوعه على الان
لا يحتاج الى مساحة بحسبه وان اراد ما واحدا فعليه ان يعود شكله ونصل
د ك يخرج الى ك وكذا ح ط ويخرج الى ل ونصل ط م ح ويخرج من د
عمود د ن على ح ط ونأخذ
جذر ضعف مربع وسعته
الطاق وهو ح ط د
ونأخذ نصف جيب ثلث
الدور وهو جيب زاوية
ط م فنقص قوسه عن
ثلث الدور بقيت زاوية
ط م ثم نضرب د ك
في نسبة المحيط الى القطر ونضرب الحاصل في زاوية ط د ونأخذ ثلث
الحاصل وهو مقدار ط ر فانه ارمسوح ثم نزيد د ك بحسب الطاق على د



حاجب لمصنفه
ارتفاع هذا الطاق
علمان وسنة واحد
سطح كسحا

It is the number written in the fifth column. If you can extract these ratios in three ways, then it would not be hard for you to do the same for the fourth way because it is easy. The radius of the two arcs of its concave part is equal to two thirds of its span, and the radius of its concave part is equal to an arc such that the sine of its complement is one eighth of the diameter. As for the volume of the vault in the fifth way, it suffices to multiply the square of its span by $[0;11:13:21]$ ⁸² thirds, or by $[187]$ ⁸³ thousandths to get the area of its hollow. We multiply it by the width of the vault and subtract the product and the volume of the hollow part underneath it from the volume of the wall, because its hanger does not count for any volume. If one does need it, then one needs to re-draw, and connect CD to extend it to K . Similarly, one needs to connect CI and extend it to L . Then, connect IM and CY and draw a line segment, DN ⁸⁴, from D perpendicular to DY ⁸⁵. We take the square root of double the square of the vault's span, which is the line segment CD . Then, we take half the sine of one eighth of the circle, which is the sine of the angle CIM . We subtract the arc of that sine from one eighth of the circle, what remains is the angle ICM . Then, we multiply CD by the ratio of the circumference to the diameter, and multiply the product by the angle ICD . We take one third of the result, which is the measurement of ID in the same unit used to measure AD . Then, we add DK , the width of the vault to CD .



⁸² There is an empty space here in this manuscript and in manuscripts [4, 5] as well. It has been completed by Rosenfeld, Segal, and Youschkevitch in [31] as 5 : 43, 28 but we chose to take the value computed by Nabulsi [6] as we believe it is more accurate.

⁸³ There is an empty space here in this manuscript and in manuscripts [4, 5] as well. It has been completed by Rosenfeld, Segal, and Youschkevitch in [31] as 933 but we chose to take the value computed by Nabulsi [6] as we believe, it is more accurate.

⁸⁴ A note indicated by a sign, related to DN , on the left margin: "A note from the scribe: The height of the vault, assuming its span is one. 1 : 19, 21, 20".

⁸⁵ According to the figure in the manuscript, it is supposed to be CY . Moreover, it is CY in manuscript [4].

ليحصل $\overline{ك}$ نصف قطر محدب الطاق ونضرب $\overline{ك}$ في نسبة
 المحيط الى البقط ونضرب الحاصل في مقدار زاوية $\overline{ط}$ $\overline{د}$ ونأخذ ثلث
 الحاصل فهو نصل قوس $\overline{ك}$ على $\overline{ط}$ $\overline{د}$ تمامه $\overline{ا}$ ممسوح زبدي نصفه على
 $\overline{ط}$ ليحصل نصف مجموع $\overline{ط}$ $\overline{ك}$ $\overline{د}$ نضربه في $\overline{ك}$ ليحصل مساحه قطعه
 خلقه نصف $\overline{ط}$ $\overline{ك}$ $\overline{د}$ ثم نقسم $\overline{ا}$ بل $\overline{ا}$ وسعة الطاق على $\overline{د}$ $\overline{ا}$ عني
 $\overline{ك}$ منخطا فما خرج بقوسه في الجيب ثم نقسم مربع $\overline{ك}$ $\overline{د}$ $\overline{ا}$ منحن الطاق
 ونزيد جذره على وسعة الطاق ونقسم المجموع على $\overline{ك}$ منخطا فما خرج
 بقوسه في الجيب فماخذ التفاضل بين القوسين فهو قوس $\overline{ك}$ تمامه
 للمحيط ثلثا $\overline{ا}$ وستين اعني زاوية $\overline{ك}$ فيحصل مقدار تمامه $\overline{ا}$
 واحد بنسب ما و نضرب $\overline{ك}$ في نصفها ليحصل مساحه قطاع $\overline{ك}$
 $\overline{د}$ ثم نضرب جيب زاوية $\overline{ك}$ في خط $\overline{د}$ منخطا ليحصل عمود $\overline{د}$
 نضربه في خط $\overline{ك}$ ليحصل مساحه مثلث $\overline{د}$ $\overline{ك}$ $\overline{ا}$ ننقصه عن قطاع $\overline{ك}$
 $\overline{د}$ $\overline{ا}$ بقى سطح $\overline{ك}$ $\overline{د}$ $\overline{ا}$ وعلى ذلك التباس ليحصل سطح $\overline{ك}$ $\overline{ط}$ $\overline{ا}$
 وبجمعها مع قطعه طلقه $\overline{ط}$ $\overline{ا}$ ليحصل سطح $\overline{ط}$ $\overline{ك}$ $\overline{ا}$ $\overline{د}$ نصف وجه الطاق
 نضرب ضعفه في عرض الطاق ليحصل مساحه مجسم الطاق ولان المحدب
 هذا الطاق لا يكون متساويا يترزايد سحنه ما اور دنا منه ابجد ول ولذلك
 جعلنا الضلعين العاليين من اللون في الوجود المتقدمه خطين مستقيمين
 ليكون متساويا فيها وهذا ما وعدنا **واما** مساحه سطح الداقل والحاج
 من الطاق اعني المنحنين نضرب عرض الطاق في مقع وجهه ليحصل مساحه

to get CK , which is the radius of the convex part of the vault. We multiply DK by the ratio of the circumference to the diameter, and multiply the product by the measurement of the angle ICD . We take one third of the result, which is the difference between the arcs KL and ID , in the same unit used to measure AD . We add its half to ID to get half the sum of ID and KL . We multiply it by DK to get the area of a portion of an annulus $IDKL$, then divide AC , or rather AD which is the span of the vault, by CY , i.e., CK , reduced. We take the arc of the result from the sine, then take half of the square of DK , the width of the vault, and add its root to the span of the vault. We divide the sum by CK reduced. We take the arcsine of the result. Then, we take the difference between the two arcs [calculated earlier], the result is the arc YK in the unit in which the circumference is three hundred and sixty. This arc is the angle YCK , which we obtain its measurement in the unit in which AD is one, analogous to what was mentioned earlier. We multiply CD ⁸⁶ by half of that angle, to get the area of the sector KCY . Then, we multiply the sine of the angle YCK by the length of the line segment CD reduced. So we get [the length of] the perpendicular line segment DN . We multiply it by the length of the segment CY to get the area of the triangle DCY . We subtract it from the sector KCY , what remains is the surface of KDY . Similarly, we obtain the surface EIL , and add it to the portion of the annulus $ILKD$ to get the surface of $IEYD$, which is half the face of the vault. We multiply its double by the width of the vault to get the volume of the vault. Since the convex part of this vault is not proportional to the increase of its width, we did not include it in the table. For this, we made the upper two sides of the right kite, in the aforementioned ways, into straight lines so that they become proportional, and this is [the explanation] that we have promised.

As for the inner and the outer surface areas of the vault, i.e., the two curves, we multiply the width of the vault by the concave part to get the area

⁸⁶ We believe it should be CK instead.

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سطح الباطن وفي محله يحصل مساحة سطح الظاهر وقد اطنبنا في
 مقاصد هذا الفصل لان الاحتياج بكثرة ولم نذكره المتقدمون على ما ينبغي
الفصل الثاني في مساحة القبة وهي ما على هيئة نصف كرة
 مجوفة واما على هيئة قطعة كرة مجوفة واما على هيئة مخروط مضلع واما
 على هيئة يحصل عن توهم ادا وجه الطاق اى طاق من الطين المذكور
 على خط ارتقاء اعني خطا وصل بين محده ومنتصف ما بين قاعدة
 اما مساحة النوعين الاولين فقد ذكرنا كيفية مساحة الكرة وقطعتها
 واما مساحة النوع الثالث فمذكورة في مساحة المحروط واما مساحة
 النوع الاخر فلمساحة سطحه نجعل قطبه مركزا ونزيد على سطحه محيطات
 ودوائر كثيرة بحيث لا يعقد التقاطعات بين الخطوط المنحنية الواحدة
 بين كل اثنين منها وبين المنتهية التي كانت تلك المنحنية واطن
 ان يكتسب بسبع او ثمانية من تلك المحيطات ثم نمنح من رأس القبة
 الى محيط كان اقرب اليه ونضرب في نصف ذلك المحيط ثم نمنح كل
 واحد من المحيطات ونمنح نصف مجموع كل متجاوزين فيما بينهما ونجمع
 حواصل ضرب ليكون مساحة سطح القبة واما مساحة مجسم فنرض
 ما بين رأس القبة ووسط الدائرة القريبة من الدوائر المرسومة عليهما
 مخروطات ما ما بين كل دائرتين من تلك الدوائر مخروطات ناقصا ونحسبها
 كما ذكرنا ونجمعها ثم نمنح مخروطات الهواء الكالية اعني مجوف القبة
 وننقصها منها فماتت فهو مساحة مجسم القبة وقد علمنا ما في القبة التي

of its inner surface. We multiply the width by the convex part to get the area of the outer surface. With this, we have fulfilled the purpose of this section, because there is a dire need for it, and the predecessors did not cover it properly.

Second Section. On the area of a dome. A dome looks like either a hollow hemisphere, or a hollow spherical cap, or a polygonal cone, or it is a result of imagining that we rotate the face of a vault, i.e., any of the aforementioned vaults, around its height, i.e., the line that connects its definer and the middle point of the distance between its two bases. As for the measurements of the first two types, we have already mentioned how to obtain the measurements of a sphere and its portion. As for the third type, it is mentioned in the measurements of a cone. As for the last type: to calculate its surface area, we make its pole a center, and draw many circles on its surface so that the distance between two consecutive curves and the straight line segments that are like chords for those curves is negligible. I believe that seven or eight of these circles would suffice. Then, we measure the distance between the vertex of the dome and the closest curve to it. We multiply this distance by half the circumference of that circle, then we measure all the other circumferences, and measure half the sum of each adjacent two. Then, we add the products [of each circumference and the shortest distance between it and its adjacent circumference] to get the surface area of the dome. As for the volume, we assume that what is between the vertex of the dome and the position of the closest circle drawn is a full cone, and the surface between each two circles a frustum. Then we measure them as we mentioned earlier and add them up. Then, we measure the empty air-filled cones, i.e., the hollow part of the dome. We subtract the empty ones from the initial ones, the remainder is the volume of the dome. We have done this in the dome that was

علمت لسم كرم مقعر الطاق بالوجه الرابع واستخرجنا نسبة المساحة
 الى مربع قطر القاعدة ليسهل منه العمل وطريقه ان نقرب مربع قطر
 مقعر قاعدة القبة في اموالب ثانياه او في ١٧٧٩ على ان اول
 مراتبه ثالث الاعشار يحصل مساحه سطح مقعر القبة ولو ضرب
 مربع قطر محذب القاعدة فيه يحصل مساحه سطح محدبها لانها غير
 متوازيين ولو ضرب كل واحد من مكعب قطر مقعر قاعدة قاعدها ومكعب
 قطر محدبها في ٢٤ في ثانياه او في ٥٤٥٣ على ان اول مراتبه ثالث
 الاعشار وناخذ التفاضل بين الحاصلين فهو مساحه مجسم القبة المجوفه

الفصل الثالث

نسبة سطح القبة الى مربع قطرها	اموالب	١٧٧٩
نسبة مجسم القبة مضاعفا الى مكعب قطرها	٤٦ ٤	٥٣٥٦

في مساحه سطح المترنس
 وهو مستقف كدرج ذات اضلاع و سطح كل ضلع منه يتقاطع مع
 ما يجاوز على زاوية اما قائمه او نصف قائمه او مجموع قائمه و
 نصف او غيرهما وبها قائمتين في الوهم على سطح مواز للافق و
 مبني على فوقهما سطح مستوي غير مواز للافق او سطحين مستويين او
 منحنيين بهما مستقيما ويقال لهما مع مستقيهما بيت واحد ويقال
 للبيت المجاوز التي قواعدها على سطح واحد مواز للافق طبقه
 واحد ويقال لمقدار قاعدة اعظم الاضلاع متباعد المترنس و
 ما شابهها فاربعه انواع المترنس الساذج الذي يدعاه النفاثون
 بيرومبه والمطين والقوس والشيرازي امت الساذج فهو

built in Sinjar. It was drawn like the concave part of the fourth form, and we extracted the ratio of the area to the square of the diameter of the base so that calculations become easier. The way to do this is to multiply the square of the diameter of the concave part of the base of the dome by 1 : 46, 32 seconds or by 1775 with the first thousandths' position we get the surface area of the concave part of the dome. If we multiply the square of the diameter of the convex part of the base by the same number, we get the surface area of the convex part because they are not parallel. If we multiply each of the cube of the diameter of the concave part of the dome's base and the cube of the convex part by 0 : 58, 28 seconds, or by 306 as the first thousandths' position, and by taking the difference between the two products, we get the volume of the hollow dome.

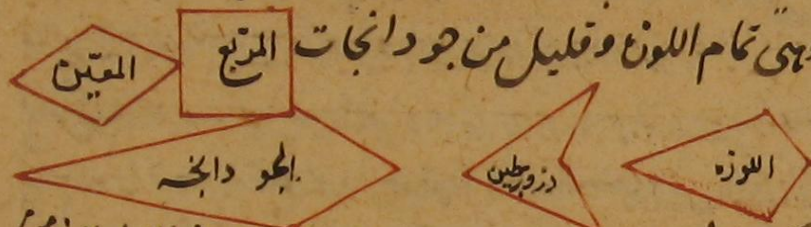
	Sexagesimal			In Indian			
	Degrees	Minutes	Seconds	Units	Tenths	Hundredths	Thousands
The ratio of the area of a dome to the square of its radius	One	Forty-six	Thirty-two	1	7	7	5
The ratio of the volume of a solid dome to the cube of its radius	Zero	Eighteen	Twenty-three	0	3	0	6

Third Section. On the surface area of the muqarnas. A muqarnas is a stair-like ceiling that has facets and a surface. Each facet intersects with its adjacent either on a right angle or half a right angle or the sum of one and a half right angles, or others. The two facets can be thought of as perpendicular to a plane parallel to the horizon. Built over these two facets is a plane not parallel to the horizon, or two planes, or two curved surfaces, which are the ceiling of the facets. The two facets along with their ceiling are called a cell. Adjacent cells with bases on the same plane parallel to the horizon are called a tier. The length of the base of the largest facet is called the module of the muqarnas. We have seen four types of muqarnas: Simple muqarnas, which is known by constructors as minbar-like muqarnas, clay-plastered, arched, and Shirazi. As for the simple muqarnas, it is

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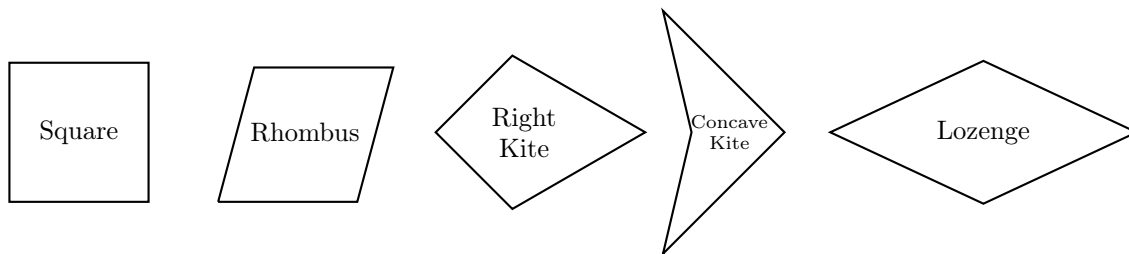
٩٥

ما يكون سطوح اضلاع بيوته معينات وشبهات بالمعين و
 مستطيلات لا غير وسطوح اعلاها اعن سقوفها مربعات و
 معينات ولوزجات وانصاف مربعات ومعينات وذوات الزوايا
 وهي تمام اللون وقليل من جود انجات المربع المعين



ويكون اضلاع المربعات والمعينات والضلعا الاطولاني من
 اللوزجات وذوات الرجلين وساقا نصف المعين والمربع و
 الضلعان الاقصران للجد انجات كلها متساوية ومساوية
 للمقياس ولا يكون الجود انجات الا على الطبقة العليا وطريق
 ان نسمي اولها بمقياسه ثم اردنا يكونها الى مقياس اخر كذراع او غيره
 وذلك ان بعد اضلاع كل طبقة كم يكون مبنيا على ضلع مربع او
 ضلع يساويه او ضلع المربع عليه وكم على احد الضلعين الاقصرين
 اللون او تمامها ان ذوات الرجلين او هو عليه وكم على
 قاعدته نصف المعين او هو عليه وتأخذ لكل ما هو على ضلع المربع
 او المعين واحد او ما هو على احد الضلعين الاقصرين للوزة و
 تمامها ٦ كد س ٢ ح رابعه او ٢١ كم ١ كم سادس الا عشر
 وما هو على قاعد نصف المعين ٦ م م نه سطره رابعه او ٣٤
 ٧ ٥ سادس الا عشر ونجمعها ونضرب للمجموع في سمك تلك الطبقة

the one whose cells' facets are rhombi, rhomboids, or rectangles only, and with upper surfaces, i.e., its ceilings, squares, rhombi, right trapezoids, halves of squares or rhombi, concave kites which are the complements of right rhombi, or some lozenges.



The sides of the squares, and rhombi, the two long sides of the right kite and the concave kites, the two legs of the halves of squares and rhombi, and the two short sides of the lozenges all have the same length and are equal to the module. The lozenges only exist on the upper tier. The way to find the measurements of a muqarnas is to first measure in terms of its module. Then, if we want, we can convert it into another unit like a cubit or any other. This is done by counting the number of facets in every tier; how many are built on a side of a square or a side that equals it, or the ones that have a side of a square built on them, and how many are on one of the two short sides of a right rhombus or its complement, i.e., the concave kite, or the ones on which the sides are built, and how many are built on the base of the half of a rhombus or the ones on which the base is built. We take one for each [facet] that is built on a side of a square or a rhombus, and for one of the two shorter sides of the right kite or its complement, 24 : 51, 10, 08 fourth or 414214 hundred-thousandth, and for what is built on one of the two short sides of a rhombus and its complement, 0 : 21, 12, 47, 22 fourth or 765367 hundred-thousandth. We add these values up, and multiply the sum by the thickness of that tier,

i.e., the height of the facets, which equals the module in most cases. What results is the area measure of that tier's facets, i.e., walls, in terms of the module of the muqarnas. Then, we take one for each square built on the ceiling, and $0;42 : 25 : 34 : 4$ fourth or 707107 hundred-thousandth for a rhombus, and $24 : 51 : 10 : 08$ fourth or 414214 hundred-thousandth for a lozenge, $21 : 12 : 47 : 32$ fourth or 352553 hundred-thousandth for a half of a rhombus, $0 : 17 : 34 : 24 : 36$ fourth or 292093 hundred-thousandth for the completion of a lozenge, and a half for a half of a square. When we add the values up, the sum is the surface area of the ceiling of the tier in terms of the module of that muqarnas. Then, we measure the area of all tiers to get the area of a muqarnas. If we measure the area of the surface on which the muqarnas is built, then we get the area of the entire ceiling of the muqarnas. Then, if we want to convert the area to cubits, we divide it by the square of what is in one cubit, similar to what is in alike used unit and its parts. What results is the desired value.

As for the clay-plastered muqarnas, we have seen it in old buildings in Isfahan. Most of them have the same appearance as a simple muqarnas, except that the heights of the tiers are not equal. It is also possible that it has two or three tiers that only have ceilings and no facets. Finding its area is similar to finding that of a simple muqarnas.

As for the arched muqarnas, it is a simple type of muqarnas whose ceilings are curved. Between the ceilings of each pair of cells, there is a curved surface in the shape of a triangle or two connected triangles like a concave kite. It is possible for curved triangles like the one mentioned to exist in some of its ceilings, with curved right rhombi or lozenges over them. The facets of the cells are either squares or rectangles,

nothing else. The bases of these facets either equal the module of the muqarnas, half the diagonal of its square, the difference between its diagonal and its side, or the length of the side of an octagon with half of its longest diagonal, the module. These four cases exhaust all the possibilities for bases. The way to calculate its area is to count the number of facets, how many of them are built on bases that equal the module, and how many are built on ones that equal half the diagonal of its square, and how many are built on what equals the difference between its diagonals and its side, and how many are built on what equals the side of an octagon with half of its longest diagonal equal the module. We take one for each facet in the first case, 0;42 : 25 : 35 : 4 fourth or 707107 hundred-thousandth for the second case, 0;24 : 51 : 10 : 8 fourth or 414214 hundred-thousandth for the third case, and 0;45 : 55 : 19 : 15 fourth or 765367 hundred-thousandth in the fourth case. We add these values and multiply the sum by 1;43 : 33 : 45 : 41 fourth or by one and 726045 hundred-thousandth to get the surface area of all the cells in terms of the module of the muqarnas. We call this number the multiplier. Then, we count the number of curved triangles or curved concave kites that are between the ceilings. We take 0;34 : 1 : 38 : 55 fourth or 567129 hundred-thousandth for each triangle, 0;36 : 37 : 10 : 57 fourth or 610328 hundred-thousandth for each small concave kite, 1;0 : 52 : 5 : 19 fourth or 14473 hundred-thousandth for each big concave kite, and 0;38 : 1 : 21 : 3 fourth or 633709 hundred-thousandth for each curved right kite. If there are Lozenges at its top, we multiply the number of times the length of the module appears in the long diagonal by half of the short diagonal. Then, we multiply the result

by the number of lozenges, then we add the surface area of the cells, the triangles, the concave kites and the right kite that occurs between the ceilings, and the lozenges to get the area of the muqarnas.

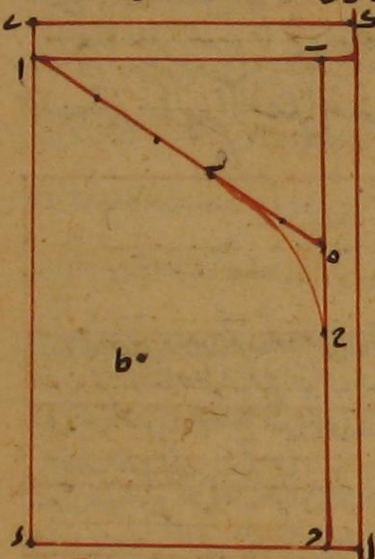
As for the Shirazi muqarnas, it is similar to an arched muqarnas, except the measurements of the bases of the cells in an arched muqarnas cannot be other than the four that we have mentioned before, while a Shirazi muqarnas has infinite possible measurements. In its ceiling, beside the curved ceilings of the cells and the triangles and concave kites between them, there are triangles, squares, pentagons, hexagons, girih, and other flat and curved surfaces. It is possible that there is a facet that does not have a ceiling in that tier, so a mihrāb is drawn on it. The way to calculate its area is to make a ruler that equals the module of the muqarnas, and divide it into small parts. It is best to divide it into sixty parts if we are calculating with sexagesimal numbers, and into ten parts if we are calculating with Indian numerals. Using this ruler, we measure the bases of the facets of all the cells in each tier except for those that have no ceiling. We multiply this value by the multiplier which is $1;43:33:45:41$ fourth or 1726045 hundred-thousandth. What results is the area of all the surfaces of the cells. Then we measure each one of the outer heights of the concave kites on its longest side. We add those heights and multiply the sum by $0;45:55:2:27$ fourth or 765290 hundred-thousandth to get the area of the concave kites. Then, we take the area of all the surfaces in the muqarnas other than the surfaces of the cells and concave kites, such as the triangles, the squares, the pentagons, the hexagons, the facets that have no ceilings, and others, using that ruler in the way that we mentioned before in calculating their areas. We add them to the areas of cells' surfaces

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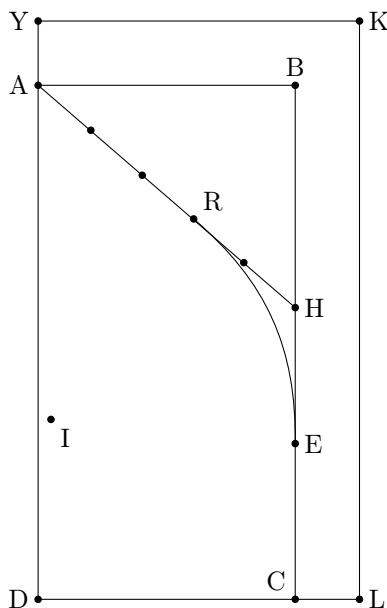
ط
البنائين

وذوات البرجلين ليحصل مساحة سطح ذلك المترنس بنفسه
 اعلم ان البنائين نرسمون مستطيلا يكون عرضه مقياس المترنس وطوله
 ضعف العرض كاستطيل ا ب د ه ونجني حون من احدى زواياه كزاوية
 امثلا ح ط ا ه بحيث يحيط مع ا ب زاوية
 مع ثلث قائمه ويسمونها ه خمسه اقسام
 ونأخذون من نقطه ه د بقدر القسمين
 منها و ه ج ايضا مثل ه ر وندير و ن على
 كل واحد من نقطتي ج ر بعد ربع قوسين
 تقاطعان داخل المستطيل على نقطتي
 ط وندير و ن على نقطتي ط قوس ربع
 فلي لا محالة يكون سدس المحيط ونخرجون
 خطا د ا ر ه على الاستقامة مقدار ايسر الى نقطتي د ه ويحيطون د ك
 موازيا ل ا ب و د ك موازيا ل ا ب ثم يعملون من ا ك خطا ك ه بحيث
 ينطبق كل واحد منها على سطح ك ه ا ر ه على ا ن ربع قوسين ويجعلون
 كل اثنين منها محيطا بيت واحد بحيث يكون ضلع ه ج منه سافولنا =
 فاستخرجوا مقدار ربع ا ر ه ج على ا ن ا ب واحد فوجدنا مستقيما
 ا ر ه د لوطا و قوس ربع ه ن د ه د و خط ه ج د ه د لوطا
 د ه مجموع ا ر ه ج د ه د لوطا و مجموع ا ر ه ج د ه نصف د ه
 د ه د ك و مجموع ه ج و نصف ا ر ه د ه د ه د ك و ذلك ما سميناه



and the concave kites to get the surface area of that muqarnas.

Addendum. Know that masons⁸⁷ draw a rectangle with width the length of the module of the muqarnas, and length double the width, like the rectangle $ABCD$. They draw from one of its angles, A for example, a line AH , so that with AB , it encloses an angle that equals one third of a right angle. They divide AH into five parts, and take a line segment HR from point H , that equals two of those parts. They also draw HE with the same length as HR . They draw two arcs around each of the points R and E with radius RE that intersect inside the rectangle at a point I . They draw the arc RE around point I . The arc will inevitably equal one sixth of the circumference. Then, they extend the line segments DA and DC by a small length to points L and Y . They draw segments LK parallel to BC , and YK parallel to AB .



Then, they use gypsum to make many panels, such that each of them is equal to the surface $KYARECL$, considering RE to be an arc. Then, they make each two panels surround one cell, such that the side CE is vertical. We extract the measurement of AR , RE , and EC , assuming AB to be one. We find the length of AR to be $0;44 : 37 : 9 : 11$, the arc RE , $0;50 : 15 : 55 : 44$, and the line segment CE , $0;57 : 38 : 43 : 14$. So, the sum of $AREC$ is $2;29 : 28 : 48 : 9$ and the sum of ARE is $1;31 : 50 : 4 : 55$. Its half is $0;45 : 55 : 2 : 27$. The sum of CE and half of ARE is $1;43 : 33 : 45 : 41$, and this is what we call

⁸⁷ A grammatical correction of the word masons is on the left margin.

the multiplier, and we have used it in the measurements. They [the masons] may shorten the stand of the panel, i.e., the line segment CE , or they may lengthen it, which would happen if they put it behind the vault -they need to do so for the panel to fit perfectly. So, in similar measurements, we need to add or remove a quantity from the multiplier equal to the one added or removed from the stand of the panel. The remainder or the sum is then used instead of the multiplier.

We have put down the quantities used in this section in a table to be accurate like this.

	Sexagesimal					In Indian					
	Degrees	Minutes	Seconds	Thirds	Fourths	Units	Tenths	Hundredths	Thousandths	Ten-Thousandths	Hundred Thousandths
If the module is one, one of the shortest sides of the right kite is like this and its area assuming the square of the mode is one	Zero	Twenty-four	Fifty-one	Ten	Eight	0	4	2	4	2	1 4
Shorter diagonal of the rhombus which is the side of an octagon with half the diagonal equal the module	Zero	Forty-five	Fifty-five	Forty-nine	Fifteen		7	6	5	3	6 7
Half the diagonal of the square of the module assuming the module is one and the area of the rhombus assuming its square is one	Zero	Twenty-two	Twenty-five	Thirty-five	Four		7	0	7	1	0 7
Half the area of the rhombus	Zero	Twenty-one	Twelve	Forty-seven	Thirty-two		3	5	3	5	5 3
Area of the complement of the right kite	Zero	Seventeen	Thirty-four	Twenty-four	Fifty-six	0	2	9	2	0	9 3
If the multiplier is multiplied by the base of each cell of the arc and the Shirazi one gets its area	One	Forty-three	Thirty-three	Forty-five	Forty-one	1	7	2	6	0	4 5
If we multiply by it the height of the exterior angle of a concave kite we get its area	Zero	Forty-five	Fifty-five	Two	Twenty-seven	0	7	6	5	2	9 0
Area of the triangle of arched muqarnas	Zero	Thirty-four	One	Thirty-eight	Fifty-five	0	5	6	7	1	2 9
Area of a small concave kite that is made of two curved triangles	Zero	Thirty-six	Thirty-seven	Ten	Fifty-six	0	6	1	0	3	2 8
Area of a big concave kite that is made of two curved triangles	One	Zero	Fifty-two	Five	Fifty-nine	1	0	1	4	4	7 3
Area of a rhomboid that is made of two curved triangles	Zero	Thirty-eight	One	Twenty-one	Three	0	6	3	3	7	0 9

Glossary

Arrow: The distance between the mid-point of an arc and its chord.

Indian numerals: According to al-Kāshī (see [9]), Indian scholars put nine famous symbols for the nine numerals in this form ٩ ٨ ٧ ٦ ٥ ٤ ٣ ٢ ١. These numbers are written in *Miftāḥ al-Ḥisab* from right to left just like Arabic. Note that these numbers are similar to what is nowadays called Persian or Farsi numerals except for number 6 which is similar to Indian numerals instead. In the translation we used what is today called Arabic numerals written here from left to right: 1 2 3 4 5 6 7 8 9.

Jummal Notation/Method: Literally meaning “arithmetic of sentences”, it is a system of representing sexagesimal numbers in which the letters of the Arabic alphabet are used as digits. It is well known and used in astronomical tables and books of calculations. The numerals of the numbers are arranged based on the letters: ا بجد هوز حطي كلمن سعفص قرشت ثخذ ضظغ which are the Arabic alphabet arranged in Abjad way. They are read from right to left as “abjad hawwaz hutti kalaman sa’fas qarashat thakhadh dhazagh”. They are twenty eight letters, nine units, nine tens, nine hundreds, and one thousand. Here are the Arabic alphabet with their numerical values.

9 = ط 8 = ح 7 = ز 6 = و 5 = ه 4 = د 3 = ج 2 = ب 1 = ا
90 = ص 80 = ف 70 = ع 60 = س 50 = ن 40 = م 30 = ل 20 = ٠٢ 10 = ي
900 = ظ 800 = ض 700 = ذ 600 = خ 500 = ث 400 = ت 300 = ش 200 = ر 100 = ق
1000 = غ

The rest of the numbers are formed from these letters. The larger number precedes the smaller. When the number of thousands repeats, its number precedes the letter غ. This is the Jummal method. The dots of ب , ج , ب , and ي are not put, and the bottom of ج is not completed, like ه, to distinguish it from ح .

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